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OPTIMAL STEADY-STATE CONTROL OF A STRING OF VEHICLES IN THE PRESENCE OF NOISE AND TIME DELAY

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled OPTIMAL STEADY-STATE CONTROL OF A STRING OF VEHICLES IN THE PRESENCE OF NOISE AND TIME DELAY submitted by SESTO CARLO VESPA in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The problem of optimally regulating the position and velocity of a string of vehicles in steady-state motion on a highway, when both vehicles and controller are subjected to additive noise and feedback time delay, is considered.

Two simpler problems are first treated : 1. regulation of a string of vehicles when random vehicle and transmission disturbances are present, and 2. regulation of a string of vehicles when random vehicle disturbances and transmission delays exist. In both these instances, the optimal controller is developed using the optimal control and estimation theory available in the literature, and its performance is evaluated by simulation on an IBM System/360 digital Computer. Optimal stochastic regulator performance is compared with that of the optimal deterministic controller subjected to noise and time delay; both, in turn, are compared with the performance of the optimal deterministic regulator where noise and time delay are absent. The importance of having exact knowledge of the amount of feedback delay on the design and performance of the designed optimal regulator is also examined.

It is shown that, by proper overall system design, not only can vehicle power plant size and passenger discomfort be significantly reduced, but, what is even more important, regulation as well can be enhanced.

ACKNOWLEDGEMENTS

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CHAPTER ONE

INTRODUCTION

ABSTRACT

A brief synopsis of past work on the automatic vehicle control problem is presented with the view to indicate some of the gaps which now exist in the theory. The scope of this thesis is described.

1.1 General

An increasing awareness that a majority of problems encountered in the highway transportation systems of the major cities in North America are, to a large extent, due to inadequate vehicle control has initiated an ever-intensifying effort on the part of many researchers to understand the dynamic characteristics of traffic flow and to develop more sophisticated techniques for control than are presently available. While schemes for complete automation of the highway system have been proposed, efforts have also been directed toward the development of schemes in which the role of the human driver is either made more effective or is completely de-emphasized. To achieve the former objective some have advocated the development of various driver aids intelligently selected to help overcome basic human deficiencies in the driving task [14, 36].[1] Others have suggested revolutionary automatic systems in which the need for the human driver is virtually eliminated [15]. More practical proposals have however expounded an evolutionary concept in which the driver aided system serves as the transitional step from present normal systems to the ultimate fully integrated automatic one [36].

The control schemes for the automatic control of a string of vehicles on a roadway have ranged from decentralized control schemes

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[1] Numbers in square brackets refer to articles in the bibliography.

in which each vehicle is directed by information obtained at the vehicle itself, to strongly centralized control schemes in which the vehicle is part of a larger information system, (i.e.) each vehicle is located and commanded from data available at the central traffic control [11]. Although the centralized system provides greater flexibility and greater overall system efficiency, it is however seriously hampered by the need for a vast communications grid and an enormous initial capital outlay. Nonetheless, increasing social demands and the needs of an economy based on continuous expansion seem to be making the development of such an automatic transportation system a real necessity.

1.2 The steady-state vehicle regulator in automatic control

The automatic control of a string of moving vehicles such as that depicted in figure 1.1 requires that complete control be exercised over the position and velocity of the individual vehicles in the string at all times. Examining the acceleration, velocity, and position of each vehicle of the string (or of the string itself when all the component vehicles are moving as a unit) in figure 1.2 [11] for an ideal excursion between some origin and destination, shows that the transitional control actions required at the initial and terminal times are separated from each other by a steady-state condition of relatively long duration where the vehicle velocity is fixed. The control law required to maintain this pre-determined

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FOR EQUAL SEPARATION BETWEEN VEHICLES IN THE STEADY-STATE,

$$\Delta k \Big|_{k=1,2,\ldots,N} = \Delta$$

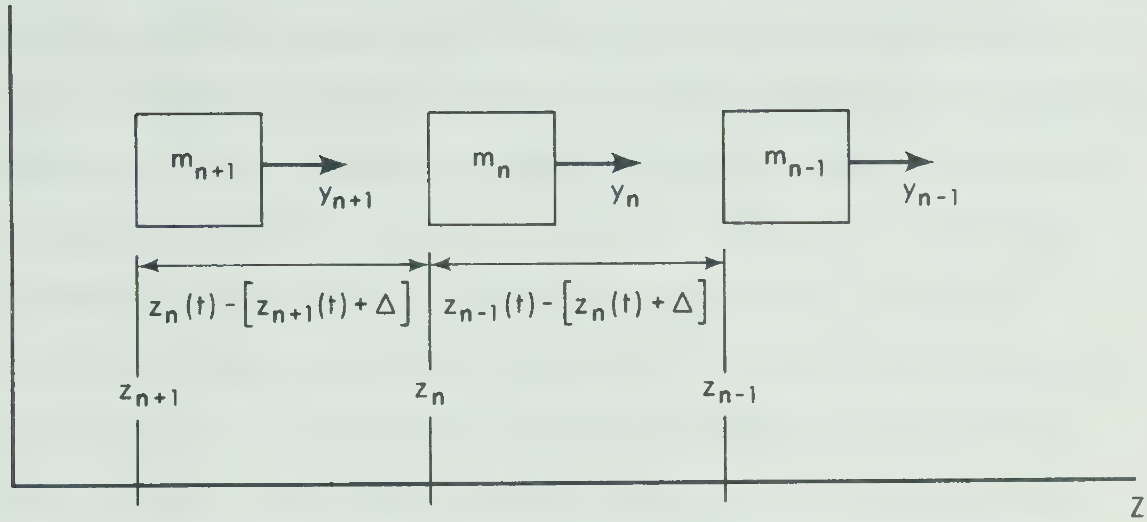

FIGURE 1.1: Three vehicles moving in a string.

steady-state condition is thus an essential element of any automatic system. It has, consequently, been the subject of intensive research for the past ten years, including that reported in this thesis.

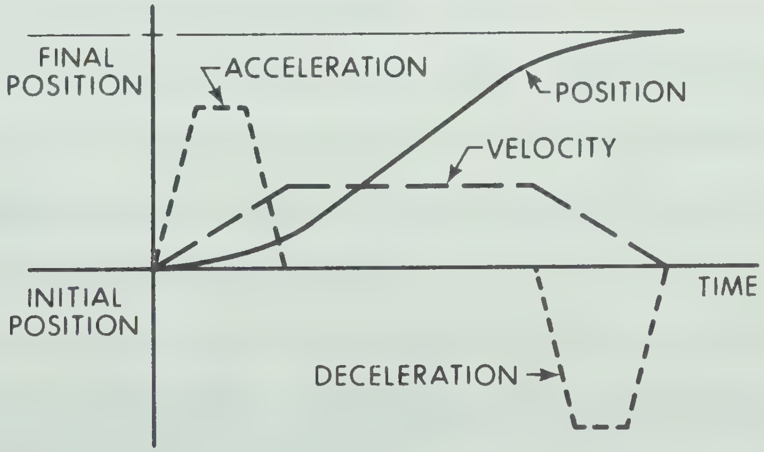

FIGURE 1.2: The acceleration, velocity, and position versus time of a given vehicle travelling between two points.

To derive the steady-state control various schemes have been proposed, with the spectrum of control theory used ranging from the classical frequency response methods to modern optimal control theory. Fenton et al [17], Fenton and Bender [7], give a clear indication of the use of classical theory in the design. From given steady-state performance specifications a linear mode controller is readily developed. A much more sophisticated and relatively recent approach to the solution of the problem involves the application of optimal control theory. The vastly increased versatility in system design that this new technique has made possible has allowed it to almost completely overshadow the classical one. Consequently, classical control techniques will not be considered at all in this present work.

1.3 Optimal control theory and steady-state vehicle control

The use of optimal control theory to regulate the position and velocity of each vehicle in a long string was first proposed by Athans and Levine [27] in 1966.[2] Given that Δ and V_0 are the desired separation distance between adjacent vehicles in the string and the desired string velocity respectively, they showed that through a suitable choice of state variables and index of performance the control problem could be reduced to the standard linear regulator problem with quadratic cost [24, 5].

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[2] Appendix one gives a short summary of the procedure and should be consulted at this point as only the implications of this scheme are considered in this present work.

The structure of the resultant optimal closed loop control system is shown in figure A1-2. From knowledge of the position and velocity of every vehicle in the string the optimal control input needed to maintain the desired vehicle separation and string velocity is determined.

Since the derived optimal control scheme requires every vehicle to have complete knowledge of the state of all other vehicles in the string continuously in time, deployment of an expensive and complex communications system is then a necessary prelude to its implementation [3]. This realization spawned intensive research into possible ways of alleviating the information processing problem without excessive degradation of system performance. What resulted from this was the development of the following two systems :

1. a suboptimal control system, and 2. a sampled data system where only samples of the state variables are transmitted every T seconds.

1.3.1 Suboptimal control

A paper by Athans, Levine, and Levis [4] expounds the basic philosophy of the suboptimal design. It proposes that each string of vehicles be considered to be constructed from a set of interlaced vehicle substrings. Having obtained the optimal control for each substring, the suboptimal control for the vehicle string can then be built up by superposition.

Some proposals have sought to divide a given vehicle string into

substrings of two or three vehicles each [37] where : a) the control for each vehicle in the string is determined only by the motion of the vehicle directly ahead, or b) the control for each vehicle is determined by the motion of the vehicle directly ahead and behind, respectively. Using a criterion proposed by R.L. Cosgriff [12], Peppard and Gourishankar [37] have found that a given vehicle string constructed from the three vehicle basic unit exhibits a greater degree of asymptotic stability than one constructed from the two vehicle one.[3]

Unlike the optimal design, the suboptimal one requires each vehicle in a long string to only have information on the position and velocity of all other vehicles in the substring of which it is a part. Simulation results [4] moreover show that the suboptimal design gives regulator performance close to that of the optimal.

Melzer and Kuo [31, 32], using the theory of generating functions, mathematically justify the philosophy of suboptimal design. For a finite string of vehicles, they found that the exact controllers for each vehicle in the string have the same structure as that for a typical vehicle in an infinite string. In fact, the individual vehicle in an infinite string is shown to so heavily weigh information from the first few adjacent vehicles that information from the

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[3] Since the linear regulator is known to be locally stable [5] all systems under consideration are at least locally stable.

remaining vehicles can be entirely ignored without incurring any noticeable change in performance.

1.3.2 Sampled data control

The possibility of alleviating the communication problem by using a sampled data analogue of the continuous controller has been studied extensively by various researchers [28, 29]. Levis [28] discusses its desirability from both an economic and technical viewpoint.

Athans and Levis [30] derived the sampled data controller and studied the dependence of regulator performance upon the length of the sampling interval. Having postulated a continuous time dynamical system with quadratic cost, they introduce sampling by constraining the control to remain constant over a specified length of time (the sampling interval). Forcing changes in the control to occur only at the sampling instants then allows them to transform the problem to an equivalent discrete time one. The discrete minimum principle [25] is then applied.

The sampled data system is shown to exhibit two modes of behaviour depending upon the closed-loop eigenvalues obtained [30]. In one mode, the behaviour of the regulator is similar to that of the optimal continuous system, whereas performance in the other mode is similar to that of an overdamped system.

Levis [28] develops a computer algorithm which allows the designer to determine the optimal control of a linear sampled data system with quadratic cost without the least knowledge of optimal control theory.

That the sampled data system can drastically cut communications requirements without great loss in regulator performance is adequately shown by Athans and Levis [30]. Choosing a sampling interval one and one half times the dominant time constant of the continuous open-loop system, they show that the optimal cost is only increased by 15%.

1.4 Need for further research

In most of the work done to date the vehicle regulator problem has almost invariably been studied under idealized conditions where no system noise and no time delay in the feedback path exist. The plant dynamics and all state variables are assumed to be known exactly while all transmission and computation delays are excluded from consideration. It is needless to point out that this ideal model does not accurately portray the interactions between vehicles and surroundings in any physical system. The optimal regulator for the ideal system could in fact be far from optimal under the non-ideal conditions existent in the physical world. In the development of an automatic controller for highway vehicles then, a more realistic model must be proposed that will allow an optimal controller to be

designed whose performance in the real world will also be optimal, or as close to the optimal as possible.

1.4.1 Effects of random disturbances

Random vehicle disturbances induced by external sources result in a random deviation of the traffic queue from the equilibrium condition [40]. The raisons - d'être of the optimal automatic system, namely larger system capacity and greater passenger safety, however require that these random deviations from equilibrium be minimized as much as possible by suitably designing the optimal controller, if they are to be even remotely attained. From an economic viewpoint Rocca [40] moreover points out that an increase in velocity and acceleration disturbances results in increased power being dissipated by the vehicles and increased passenger discomfort and vehicle wear, respectively.[4]

Vehicle dynamics have generally been formulated from the straightforward application of Newton's laws of motion where all external vehicle disturbances such as wind and road conditions have been ignored. No lengthy elaboration is thus required to show that the fidelity with which the ideal model dynamics represent the physical system is far from satisfactory.

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[4] Since the power level is proportional to the product of thrust and speed, we have that

$$d(\text{power level}) \;\alpha\; \text{thrust} \cdot d(\text{speed}) + d(\text{thrust}) \cdot \text{speed},$$

where $d(\cdot)$ denotes the operation of taking the total derivative. Because of the high speeds at which the automatic vehicles are expected to operate, these considerations are thus not altogether negligible.

If for the moment we consider the dynamics of the car-following problem to be known exactly, and if furthermore we disregard all external vehicle disturbances, ideal conditions still do not exist. Random vehicle disturbances still occur due to the existence of noise sources in the vehicle control system itself. The state measurement device,[5] the transmission system, and the controller are all sources of random errors. Even though they may not be of large magnitude, these disturbances affect each vehicle in a controlled string. Relatively large random disturbances can also originate in a particular vehicle of the string. The queue response to such disturbances is determined entirely by the closed loop characteristics of the vehicle controller [40].

An area of research where several aspects need further investigation can be described as follows :

Given a vehicle string subjected to external disturbances, and given that random errors do occur in the measurement of variables and in the transmission of information about the position and velocity of each vehicle, then how can an optimal controller be designed and how does this controller perform compared with that for an ideal system (under similar simulated real world conditions)?

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[5] If a sampled data system were being considered quantization errors in the measurement of velocity and headway could also be included here.

The digital computer control of a string of moving vehicles when random errors occur in the information exchange between vehicles and central computer has been studied by Anderson and Powner [1]. However, their work mainly consists of a qualitative analysis of observed results. (The definition of some appropriate measure of vehicle performance could thus be quite helpful here in determining the extent to vehicle performance can be improved.) They fail to consider vehicle response when the noise statistics are not accurately known. Nor have they considered the case where state measurement and control are not coincident in time.

Rocca [40] considers the problem of regulating a string of moving vehicles when vehicle disturbances occur as a result of environmental factors and system anomalies. He does however approach the problem from a classical point of view which does not yield the same kind of answers which can be expected from an optimal control formulation of the problem.[6] His analysis of vehicle disturbances and their sources will nevertheless be of great help and of practical use in the formulation of the optimal control problem.

1.4.2 Effects of feedback time delays

Another area of investigation where some problems require effort is related to time delays in the system. Careful examination shows

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[6] This paper was published the year preceding the introduction in the literature of optimal control theory to the solution of the problem.

that there exists a time delay in the feedback loop of the vehicle regulator for two reasons : 1. there is a transmission delay in sending and processing information on vehicle states from one vehicle to another, and 2. there is a finite computation time required to calculate the optimal control from a given set of state variables. Though at first glance these time delays do not appear to be significant enough to noticeably affect regulator performance, they could however be of significance in determining the maximum number of vehicles which a central processing system can handle with safety at one time. If a controller capable of reducing the effects of time delays on vehicle performance could be designed, then the versatility and ultimate capacity of the system could be increased, at least in theory (other things being equal).

To the best knowledge of the author, few papers exist which deal with the effects of time delays on the performance of vehicle strings. Whatever pertinent references were found have generally treated the problem of studying vehicle behaviour when a time delay is introduced into the feedback loop of a system designed on the assumption that no time delay exists [37]. While this approach does serve to give a rough estimate of the severity with which feedback delays affect regulator performance, there is a need for more refined techniques in the solution of the problem.

The forementioned approach does serve to further understanding of vehicle regulator problems, however, the system designed in that

way is not the optimal one when the time delay is present. Hence, it fails to give a clear indication of how well the optimal regulator can perform in the presence of feedback delays.

1.5 Scope of this thesis

The aim of the work reported in this thesis was to examine the effects of both random disturbances and feedback time delays on the design and performance of an optimal steady-state controller for vehicle strings. Digital computer simulations of the designed regulators were done on the IBM System 360 Continuous System Modeling Program (S/360 CSMP) followed by a comparison of the relative performance of each. The IBM User's Manual [20] describes S/360 CSMP as a "problem oriented program designed to facilitate the digital simulation of continuous processes on large digital machines". In practice some type of sampled data scheme will most likely be used such as that described by Athans and Levis [30]. A continuous formulation was however convenient here and, as noted earlier, when the sampled data system is operating properly its performance is similar to that of the optimal continuous system.

A brief description of the format employed in reporting this work may now be in order.

Chapters two and three each deal with a portion of the overall problem of securing the optimal vehicle control in the presence of both noise and time delay. Chapter four, on the other hand, welds together the results of the preceding two chapters and examines the total problem as initially laid down. Chapter five makes some

concluding remarks based on the work reported in the preceding four chapters and gives some suggestions for future research possibilities.

In appendix one an attempt is made to give the reader a short and basic introduction to optimal regulator theory as applied to the car following problem. Reference is made to Athans and Levine's original work on the derivation of a deterministic model and its adaption to solution using optimal control theory.

Appendix two and appendix three are provided as a short review of theory employed in the body of the thesis; the certainty equivalence property of optimal control, and the derivation of the steady-state Kalman filter, respectively.

Appendix four and appendix five serve to give the reader some understanding of the workings of the IBM System/360 CSMP program and of a few of the problems which arose in the course of simulating the models described in this thesis.

CHAPTER TWO

STEADY-STATE OPTIMAL CONTROL OF A STRING OF VEHICLES SUBJECT TO RANDOM DISTURBANCES

ABSTRACT

This chapter begins with the derivation of a stochastic model for a vehicle string subjected to various random disturbances. An optimal feedback system based on Kalman filtering and linear regulator theory is developed using results available in the literature. The various disturbances which affect vehicle performance are discussed as well as their amenability to representation by the derived stochastic model. An investigation of the random generator used to model the physical disturbances is also carried out to see how well the simulated random disturbances fulfill the requirements of the theory used. A quantitative measure of regulator performance is developed to permit a comparison of the performance of the optimal and the non-optimal stochastic regulator with that of the optimal noiseless regulator. The optimal system is simulated using the IBM System/360 CSMP program. A general discussion of observed results and their implications closes the chapter.

2.1 A stochastic model for a vehicle string

The state space representation of a three vehicle string in longitudinal steady-state motion is given in appendix one. The equations (A1-3a) and (A1-3b) are repeated here for convenience.

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}_1(t)\ ;\ \underline{x}(t_0) = \underline{x}_0 \tag{2-1a}$$

$$\underline{y}(t) = C\,\underline{x}(t) \tag{2-1b}$$

where $\underline{x}(t)$ (the state vector) εR^n, $\underline{u}_1(t)$ (the control vector) εR^m, $\underline{y}(t)$ (the measured output vector) εR^r and ; A, B, C are n x n, n x m, and r x n matrices, respectively. The (real) vector space dimensions n, m, and r are related to the number of vehicles in the controlled string, N, by

$$n = 2N-1\ ;\ m = N\ ;\ r = n.$$

In order to develop a more accurate model for the vehicle string, three additional factors will be included : 1. transmission errors in communicating information to and from the controlled vehicles, 2. controller noise, and 3. external disturbances such as wind and road conditions.

Controller noise $\underline{w}_1(t)$ and transmission noise $\underline{w}_2(t)$ (from the system control centre to the vehicles) can be easily introduced into the model by assuming that the output of the controller, $\underline{u}_1(t)$, is the sum of a deterministic signal, $\underline{u}(t)$, and the two random noise terms. In other words,

$$\underline{u}_1(t) = \underline{u}(t) + \underline{w}_1(t) + \underline{w}_2(t). \tag{2-1c}$$

Random external vehicle disturbances due to wind and road conditions can now be included by adding one further term, $\underline{w}_3(t)$. Equation (2-1a) becomes

$$\underline{\dot{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) + B[\underline{w}_1(t) + \underline{w}_2(t)] + \underline{w}_3(t). \tag{2-1d}$$

Defining an equivalent noise term $\underline{w}(t)$ such that

$$\underline{w}(t) = B[\underline{w}_1(t) + \underline{w}_2(t)] + \underline{w}_3(t) \tag{2-1e}$$

then simplifies equation (2-1d) to

$$\underline{\dot{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) + \underline{w}(t)\ ;\ \underline{x}(t_0) = \underline{x}_0\ . \tag{2-2a}$$

In a similar manner, the noise in the measurement of $\underline{x}(t)$ and

the noise introduced in the transmission of information about the states from the vehicles to the controller are taken into account by modifying equation (2-1b) as follows :

Let $\underline{v}_1(t)$ be the measurement noise and let $\underline{v}_2(t)$ be the transmission noise described above. Also, let

$$\underline{v}(t) = \underline{v}_1(t) + \underline{v}_2(t) \quad . \tag{2-2b}$$

Rewrite equation (2-1b) as

$$\underline{y}(t) = C\,\underline{x}(t) + \underline{v}(t) \quad . \tag{2-2c}$$

The resultant stochastic system will then be taken to be described by the state-output equation

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) + \underline{w}(t) \;;\; \underline{x}(t_0) = \underline{x}_0 \tag{2-3a}$$

$$\underline{y}(t) = C\,\underline{x}(t) + \underline{v}(t) \tag{2-3b}$$

where, for conveneince, $\{\underline{w}(t)\}$ and $\{\underline{v}(t)\}$ are henceforth referred to as the plant and measurement noise, respectively.

The deterministic system described by the pairs (A,B) and (A,C) is assumed completely controllable and observable (appendix

three discusses the importance of these assumptions). The noise processes $\{\underline{w}(t)\}$, $\{\underline{v}(t)\}$ are assumed to be stationary independent white Gaussian with autocovariances [23]

$$\mathrm{cov}\ [\underline{w}(t),\ \underline{w}'\ (\tau)] = W\ \delta(t-\tau)\ ;\ W > 0 \tag{2-4a}$$

$$\mathrm{cov}\ [\underline{v}(t),\ \underline{v}'\ (\tau)] = V\ \delta(t-\tau)\ ;\ V > 0 \tag{2-4b}$$

and

$$E\ \{\underline{w}(t)\} = E\ \{\underline{v}(t)\} = \underline{0} \tag{2-4c}$$

The foregoing assumptions are required to enable a solution of the problem with the available theory. Section 2.5 examines actual vehicle disturbances to see how valid these assumptions are and to see how any inconsistencies affect the model.

2.2 The deterministic regulator

Considering the optimal deterministic (no noise) regulator derived in appendix one and presented in figure A1.2 in light of the discussion of section 2.1, it can be seen that a more realistic representation of the operation of that regulator under actual operating conditions is given in figure 2.1.

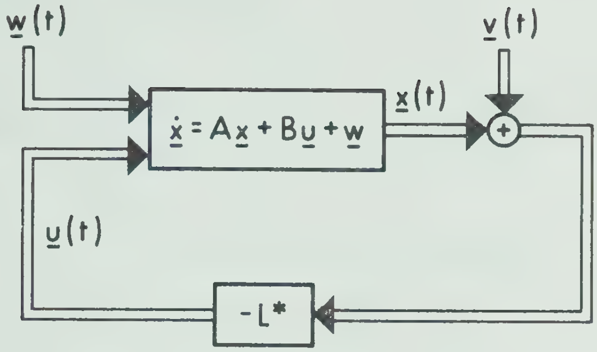

FIGURE 2.1: The deterministic regulator corrupted by plant and measurement noise (N O S R).

The presence of the noise sources $\{\underline{w}(t)\}$ and $\{\underline{v}(t)\}$ is emphasized there. That controller, for reasons to be considered in section 2.3, is now no longer the optimal one, however, and some other design must be proposed. In subsequent discussions this regulator will also be referred to as the NOSR (the non-optimal stochastic regulator).

2.3 A cost functional for the stochastic model

Because of the stochastic nature of the model developed in section 2.1, it makes no practical sense here to propose an index of performance, $J(\underline{u})$, such as given by (A1-4). The state variables $\underline{x}(t)$ and the controls $\underline{u}(t)$ are both random vectors whereupon the cost function $J(\underline{u})$ is a random variable. As a result, the optimal control $\underline{u}^*(t)$ for some specified sample function of each of the

random vectors $\underline{w}(t)$ and $\underline{v}(t)$ may not be optimal for some other sample function of each. A logical choice of performance index in this stochastic case is rather the expected value of the random variable $J(\underline{u})$, defined by

$$J_E(\underline{u}) = E\left\{\lim_{T\to\infty} \frac{1}{T} \int_0^T (x'\ Q\ \underline{x} + \underline{u}'R\ \underline{u})\ dt\right\} \tag{2-5}$$

where $R > 0$, $Q > 0$, and $E\{\cdot\}$is the expectation taken over all underlying random quantities. The term $1/T$ is inserted to keep the cost $J_E(\underline{u})$ finite as the terminal time T approaches infinity [42]. In a sense (2-5) seeks to optimize the average performance of the vehicle system (2-3).

2.4 The optimal stochastic regulator

The optimal control problem requires that, from the set of admissable controls $\underline{u}(t)$, the optimal control $\underline{u}^*(t)$ be found which will minimize the cost (2-5_ subject to the dynamic constraints (2-3).

2.4.1 Optimal control and the separation property

Due to the presence of the control vector $\underline{u}(t)$ in (2-5), minimization of $J_E(\underline{u})$ requires that the effects of the stochastic disturbances on the feedback controls first be known. In appendix two it is shown that given

$$\underline{y}(t) = \underline{x}(t) + \underline{v}(t)$$

as the measured output vector, it is generally not possible to deduce the stochastic effects of the controls $\underline{u}(t)$ since the system state vector $\underline{x}(t)$ is not known. In order to circumvent this difficulty, it is also shown there that one can rather find the conditional stochastic effects of future control actions by treating the conditional density of the state $\underline{x}(t)$ as one would the actual state $\underline{x}(t)$. The Gaussian assumptions of section 2.1 then allows parametrization of the conditional density (which is of infinite dimension) by its conditional mean and covariance, each in a finite dimensional space [42]. If the covariance is independent of control and observation (see appendix two), then only controls of the form

$$\underline{u}(t) = \phi(t, \hat{\underline{x}}(t)) \text{ for some } \phi(\cdot,\cdot) \tag{2-6}$$

need be sought. The process being controlled is now the conditional mean process $\hat{\underline{x}}(t)$.

A rather obvious conclusion that can be drawn from an examination of equation (2-6) is that the estimation problem and the control problem can be separated if the state estimator $\hat{\underline{x}}(t)$ can be designed independently of any control considerations. Under this condition, the certainty equivalence property, discussed in appendix two and applied in equation (2-6), is often termed the separation property.[7]

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[7] For a rigorous proof see Tse [42], Kleinman [23], or Wonham [44].

In the case where the separation property holds, the optimal control for the cost function (2-5) can be obtained in two steps : 1. find the conditional mean estimate $\hat{\underline{x}}(t)$ of the current state, and 2. find the optimal feedback gains, L*, treating the conditional mean estimate as the true state of the system. Since the optimal feedback gains are found by assuming that the conditional mean estimate $\hat{\underline{x}}(t)$ is the true state of the system, it is obvious that, given the cost function (2-5), the optimal feedback gain matrix, L*, of the stochastic system is identical to that of the deterministic one of appendix one.[8] Hence

$$\underline{u}^*(t) = -R^{-1}B'\hat{K}\,\hat{\underline{x}}(t) = -L^*\,\hat{\underline{x}}(t) \tag{2-7}$$

It should be noted, however, that the separation property is rather a coincidental result of the theory (i.e. that the estimator $\hat{\underline{x}}(t)$ can be designed independently of the control $\underline{u}(t)$) and does not hold in the general case of nonlinear systems where control and estimation are interrelated.

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[8]Note that the difference in a factor of 1/2 between equations (2-5) and (A1-4) is of no importance since only the relative value of the weighting matrices Q and R determine the final answer obtained.

2.4.2 The Kalman filter in the control loop

For the constant system described by equations (2-3), it is shown in appendix three that, invoking the assumptions of complete controllability and observability as well as the stationarity of the disturbance noises, the steady-state Kalman filter (modified to include the effects of the deterministic input $\underline{u}(t)$) can be derived. This steady-state Kalman filter, given by

$$\dot{\hat{\underline{x}}}(t) = A\hat{\underline{x}}(t) + \Sigma_\infty C'V^{-1}(\underline{y}(t) - C\hat{\underline{x}}(t)) + B\underline{u}(t) \tag{2-8}$$

where Σ_∞ (the conditional covariance of the state) is the unique solution to the algebraic Riccati equation

$$A\,\Sigma_\infty + \Sigma_\infty A' - \Sigma_\infty C'V^{-1}C\,\Sigma_\infty + W = 0\ ;\ \Sigma_\infty > 0 \quad , \tag{2-9}$$

is the best linear estimator of the state of the completely controllable and completely observable constant system (2-3), in terms of the output process $y(\cdot)$ over the time interval $(-\infty, t)$. Design of the Kalman filter rests solely in the choice of Σ_∞ (the steady-state conditional covariance of the state); which is done independently of any control considerations. Recalling the discussion in subsection 2.4.1 on the separation property, it is then evident that in this instance the problem of control and estimation are separable.

The optimal controller in the presence of driving noise and measurement noise thus requires: 1. the Kalman filter estimate of the system states, and 2. the optimal feedback gains (obtained as in the deterministic case of appendix one) to operate on the state estimate to give the optimal system control $\underline{u}^*(t)$. The resulting optimal stochastic regulator (also to be referred to as the OSR) is shown in figure 2.2.

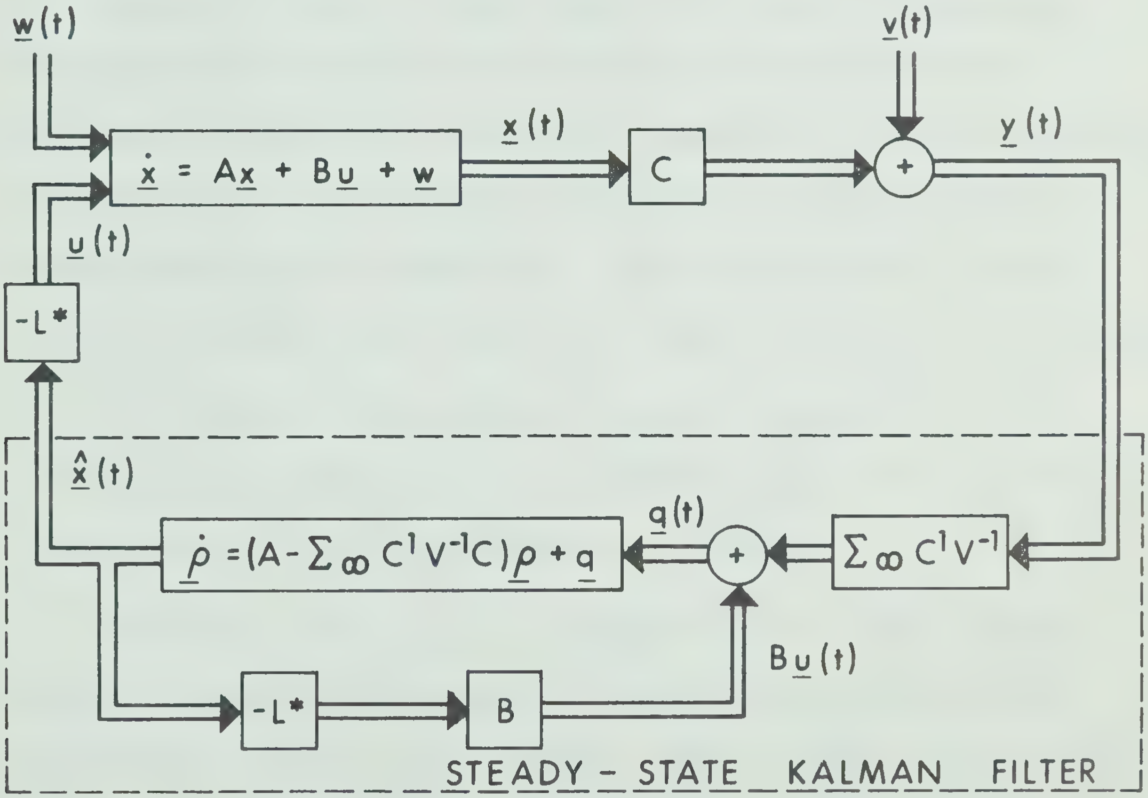

FIGURE 2.2: The optimal stochastic regulator (OSR).[9]

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9 To permit the steady-state Kalman filter of equation (2-12) to be drawn in the block diagram form shown in the figure, recall that $\underline{y}(t) = C\underline{x}(t)$.
Rearranging (2-12), after substituting for $\underline{y}(t)$, gives
$\hat{\underline{x}}\,(t)=[A-\Sigma_\infty C'V^{-1}C]\hat{\underline{x}}(t)+B\underline{u}(t)+\Sigma_\infty C'V^{-1}\underline{y}(t)$

2.5 Driving and measurement noise covariance matrices

Design of the steady-state Kalman filter requires specification of the driving noise and measurement noise covariances as

$$E\,\{\underline{w}(t)\;\underline{w}'(\tau)\} = W\;\;\delta(t-\tau)$$

$$E\,\{\underline{v}(t)\;v'(\tau)\} = V\;\;\delta(t-\tau);\; V > 0.$$

Actually, noise is never exactly white and, in many cases, the assumption that the noise is white is quite inappropriate. Frequently, however, the Gaussian noise added on to a desired signal has a relatively flat spectrum with components that extend well beyond those that are significant in the signal itself. In these cases the assumption that the noise is Gaussian and white is quite valid.

To judge the reasonableness of such an assumption in the vehicle control problem under consideration here, a brief examination of the various vehicle disturbances is in order.

Controller noise $\{\underline{w}_1(t)\}$ is essentially due to the thermal noise of the controller components (which is proportional to the temperature) and the noise associated with the vehicle state measurement device. Assuming that the temperature and vehicle velocity remain relatively constant then this noise source is essentially stationary and the white noise assumption holds fairly well [40],

[9].[10] In any modern communication system however, the signal output of the receiver is generally well above the background noise of the system so that $\{\underline{w}_1(t)\}$ can be ignored for all practical purposes. Nevertheless, occasions do arise when large electrical disturbances (both natural and man-made) do cause a significant deterioration of the received signal. Since the pass-band of the optimal regulator is rather narrow, these relatively large disturbances, $\{\underline{w}_2(t)\}$ and $\{\underline{v}(t)\}$, can generally be accomodated as white noise for purposes of this model. Moreover, it would seem that they may be approximated as wide-sense stationary processes which, as a result of the Gaussian assumption, imply strict sense stationarity. Of course these latter statements on $\{\underline{w}_2(t)\}$and $\{\underline{v}(t)\}$ are not strictly justifiable. The wind and road disturbances$\{\underline{w}_3(t)\}$will also be taken as stationary white Gaussian noises even though this assumption is somewhat less tenable than it is in the case of $\{\underline{w}_2(t)\}$ and $\{\underline{v}(t)\}$.

Further, assuming for purposes of analysis that the components of$\{\underline{w}(t)\}$and$\{\underline{v}(t)\}$are independent of each other, the covariance matrices are readily written down as

---

[10] A strict-sense stationary random process is defined as one for which all density functions are independent of absolute time reference (time origin) [45].

$$W = \begin{bmatrix} \sigma^2_{w_1} & & & & \bigcirc \\ & \sigma^2_{w_2} & & & \\ & & \sigma^2_{w_3} & & \\ & & & \sigma^2_{w_4} & \\ \bigcirc & & & & \sigma^2_{w_5} \end{bmatrix}$$

$$V = \begin{bmatrix} \sigma^2_{v_1} & & & & \bigcirc \\ & \sigma^2_{v_2} & & & \\ & & \sigma^2_{v_3} & & \\ & & & \sigma^2_{v_4} & \\ \bigcirc & & & & \sigma^2_{v_5} \end{bmatrix}$$

where

$$E[w_i w_j]_{i \neq j} = E[v_i v_j]_{i \neq j} = 0 \quad ; \; i,j=1,2\ldots,5$$

$$E[w_i^2] = \sigma^2_{w_i} \;,\; E[v_i^2] = \sigma^2_{v_i} \; ; \; i=1,2\ldots,5$$

and σ^2 is the variance.

With no loss in generality, assume unit noise variance for all noise sources. Thus,

$$\sigma^2_N \triangleq \sigma^2_{w_i} = \sigma^2_{v_i} = 1 \quad ; \quad i = 1,2\ldots,5$$

and covariance matrices W and V then become

$$W=V=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2-10}$$

2.6 Simulation studies of a typical system

Having specified the three-vehicle stochastic model (2-3) with the observation matrix C set equal to the identity matrix and with the autocovariance matrices W and V as given by (2-10), the optimal steady-state stochastic control system will now be designed for the case where the performance index $J_E(\underline{u})$ is given by

$$J_E(\underline{u}) = E\left\{ \lim_{T\to\infty} \frac{1}{T}\int_0^T [10(\delta w_1^2(t)+\delta w_2^2(t)) + \delta f_1^2(t) + \delta f_2^2(t) + \delta f_3^2(t)]\, dt \right\}^{11} \tag{2-11}$$

From subsection 2.4.2, it is known that the optimal system is specified completely (and uniquely) by the steady-state covariance

---

11 Note that this performance index is a direct extension of the one given for the three vehicle deterministic regulator of appendix one (except for the constant factor of 1/2 in front of the integral sign). This should thus allow easy comparison of stochastic and deterministic regulator design.

matrix Σ_∞ of the filter and by the optimal feedback gain matrix L*. From the steady-state solution of equation (A3-6) with $\Sigma(0)=0$ (or from equation (A3-7)), Σ_∞ is found to be

$$\Sigma_\infty = \begin{bmatrix} 0.406 & 0.151 & 0.007 & 0.008 & 0.001 \\ 0.151 & 1.239 & -0.143 & -0.102 & -0.008 \\ 0.007 & -0.143 & 0.406 & 0.143 & 0.007 \\ 0.008 & -0.102 & 0.143 & 1.239 & -0.151 \\ 0.001 & -0.008 & 0.007 & -0.151 & 0.406 \end{bmatrix} \tag{2-12}$$

The optimal feedback gain matrix is (from subsection 2.4.1) identical to the one derived for the deterministic regulator of appendix one. Hence,

$$\underline{u}^*(t) = -R^{-1}B'\hat{K}\,\hat{\underline{x}}(t) = -\,L^*\hat{\underline{x}}(t) \tag{2-13}$$

where $\hat{K}$ is given by (A1-6).

The resultant system to be simulated is thus as shown in figure 2.3 (which is a specialized version of figure 2.2) with L* and Σ_∞ given by (2-13) and (2-12), respectively.

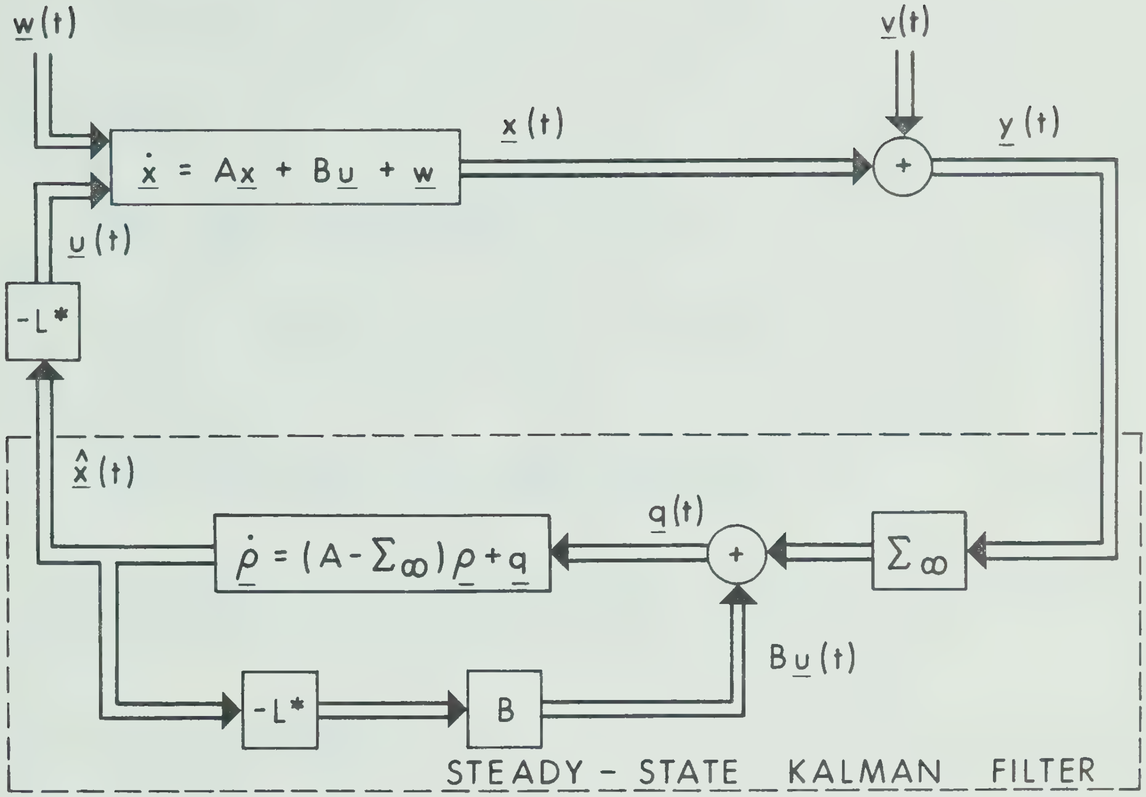

FIGURE 2-3: Block diagram of the simulated optimal stochastic regulator (OSR).

Let us now briefly see what sort of effects the inclusion of a Kalman filter in the feedback path is expected to have. Separating the deterministic and stochastic components of $\underline{x}$ and $\hat{\underline{x}}$,

$$\underline{x} = \underline{x}_D + \underline{x}_{SK}$$

$$\hat{\underline{x}} = \underline{x}_D + \underline{x}_{SK} + \delta_N = \underline{x}_D + \underline{x}_S$$

and substituting into the equation

$$\underline{\dot{x}} = A\,\underline{x} + B\,\underline{u} + \underline{w}$$

then gives

$$(\underline{\dot{x}}_D + \underline{\dot{x}}_{SK}) = A(\underline{x}_D + \underline{x}_{SK}) + B\,\underline{u} + \underline{w}$$

Now,

$$\underline{u}^*(t) = -\,L^*\,\underline{\hat{x}}(t) = -\,L^*[\underline{x}_D(t) + \underline{x}_S(t)]$$

whence

$$[\underline{\dot{x}}_D - (A-BL^*)\underline{x}_D] + \underline{\dot{x}}_{SK} = A\underline{x}_{SK} + B[-L^*\underline{x}_S] + \underline{w} \qquad (2\text{-}14)$$

Since the deterministic component evolves according to

$$\underline{\dot{x}}_D = (A-BL^*)\,\underline{x}_D$$

then,subtracting this from (2-14) gives

$$\underline{\dot{x}}_{SK} = (A-BL^*)\,\underline{x}_{SK} + (\underline{w} - BL^*\ \delta_N) \qquad (2\text{-}15)$$

where

$$\delta_N = (\underline{\hat{x}} - \underline{x})$$

is the estimation error.

For the system where no Kalman filter is present it can similarly be shown that

$$\underline{\dot{x}}_{SN} = (A-BL^*)\,\underline{x}_{SN} + (\underline{w} - BL^*\,\underline{v}) \qquad (2\text{-}16)$$

where $\underline{x}$ is now defined as

$$\underline{x} = \underline{x}_D + \underline{x}_{SN}$$

Comparing equation (2-16) with equation (2-15), it is seen that if the Kalman filter is working properly, implying that $||\underline{\delta}_N|| << ||\underline{v}||$, then the filter can very effectively reduce the stochastic effects due to the measurement noise vector $\underline{v}(t)$.

2.6.1 Simulation of system disturbances

In evaluating the effectiveness (through simulation studies) of the Kalman filter in improving vehicle performance, it is important to first of all see how well the simulated system realizes the mathematical one. The use of a digital computer to simulate the mathematical models presented, among the usual difficulties associated with digital integration techniques, added problems connected with the simulation of system disturbances. Taking into consideration the technique used for generating a random number in the IBM System/360 CSMP program, it remains to be shown that the simulated noises do fulfill the requirements of the mathematical model (e.g. with respect to mean, variance, independence, and white Gaussian assumptions).[12]

---

[12] In appendix two a brief indication of the workings of the IBM/System 360 CSMP is given. A short description of several problems encountered in the course of simulating the various mathematical models developed in this thesis is also provided.

Table 2-1 gives some data concerning the mean, standard deviation, minimum and maximum values for ten Gaussian noise sources used in this thesis. The zero mean and unit variance required of each noise source is seen to be fairly well satisfied.

To ascertain if the noise sequences are independent, a correlation analysis is carried out for every possible combination of the ten random sequences (taken two at a time).[13] Define the correlation coefficient of two random variables x and y as

$$\rho_{xy} = \frac{C_{xy}}{\sigma_x \sigma_y} \quad ; -1 \leq \rho_{xy} \leq 1 \tag{2-17}$$

where C_{xy} is the covariance between x and y, and σ_x, σ_y are the standard deviations of x and y respectively. The two random variables are said to be uncorrelated if the correlation coefficient is zero.

Table 2-2 gives the correlation coefficients for each pair of noise sequences described in table 2-1.[14] It can be seen that (in

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[13] Since the noise generator, GAUSS(·,·,·), produces a sequence of numbers having a Gaussian distribution two random variables which are mutually uncorrelated are also independent [6].

[14] Results were obtained using the "Simple Correlation and Plotting Package" (CS022) available at the University of Alberta Computing Center.

TABLE 2-1

Ten simulation noise sources*

SOURCE	MEAN	STD. DEV.	MIN.VALUE	MAX.VALUE
1	-0.0124	0.9980	-3.2333	3.1473
2	-0.0146	0.9712	-3.2111	2.8653
3	0.0374	0.9936	-2.6227	3.6630
4	-0.0099	0.9925	-4.1666	3.4578
5	-0.0018	1.0135	-2.8523	2.8727
6	-0.0545	1.0156	-3.1221	3.0076
7	-0.0675	0.9487	-3.0999	2.4462
8	0.0466	0.9398	-2.6288	3.0008
9	-0.0391	0.9960	-3.0470	3.0141
10	-0.0588	0.9971	-3.7377	2.8905

* These estimates are based on a finite data record consisting of 601 points for each of the ten noise sequences.

TABLE 2-2

Noise source correlation coefficients

	1	2	3	4	5	6	7	8	9
2	0.093								
3	0.078	-0.036							
4	-0.024	0.018	0.020						
5	0.057	-0.048	-0.007	0.028					
6	0.017	0.025	-0.034	-0.037	-0.037				
7	-0.029	0.079	0.047	0.071	0.030	0.013			
8	-0.006	0.014	-0.048	0.037	-0.020	-0.017	0.021		
9	0.025	-0.059	-0.036	-0.038	0.048	-0.004	0.022	0.001	
10	0.020	0.007	0.011	0.013	0.026	0.017	0.068	0.007	0.013

magnitude) the largest correlation, 0.093, occurs between random generators 1 and 2. Thus, although some correlation between souces does exist the degree of correlation is not significant and can be dismissed.

For each noise source present in the modeled system, the CSMP program calls subroutine GAUSS once at each iteration cycle, whereupon a single random number is generated for each integration interval. Given the integration interval, the noise cutoff frequency is then specified automatically as

$$f_c = \frac{1}{2(\Delta T)} \tag{2-18}$$

where ΔT is the integration interval specified for the CSMP integration routine.[15] Taking the integration interval to be 0.01 second (see appendix four), the cutoff frequency of the noises is then given by (2-18) as 50 Hertz.

Sectioning a single data record into twenty non-overlapping sections of 1024 data points each, and using a method described by P.D. Welch [43], an estimate of the power spectra of the random

---

[15] Note that the use of a digital computer for noise simulation has eliminated the aliasing problems encountered in digitizing continuous data. It must be borne in mind however, that given the fixed integration interval ΔT, no frequency component at a higher frequency than f_c can be modeled adequately.

number generator GAUSS is obtained.[16] The resulting power spectral estimate, normalized to its maximum value, is shown in figure 2.4. The interval from -15.54 decibels to 3.33 decibels includes the true spectrum with at least 90% certainty. Evidently, the random number generator does not, by any means, approximate the ideal white noise source. Spurious peaks (ranging from a maximum of 0 decibels to a minimum of -12.75 decibels) occur throughout the frequency spectrum.[17] The peaks do seem to occur rather uniformly throughout the spectrum however. Moreover, a variation of approximately -13 decibels between the maximum and minimum values cannot certainly destroy its usefulness as a white noise source.

In conclusion then, it seems that the random number generator GAUSS is a sufficiently good representation of the noise source postulated by the mathematical model.

2.6.2 Evaluating the performance of the optimal stochastic regulator

It is useful to evaluate the performance of the optimal stochastic regulator. An examination of equation (2-5) shows that the objective

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[16]Quoting P.D. Welch, "The method involves sectioning the record and averaging modified periodograms of the sections". The Fortran program used here is given in appendix five.

[17]Note that these spurious peaks could have been somewhat further smoothed out by averaging over a larger number of modified periodograms than the twenty used here.

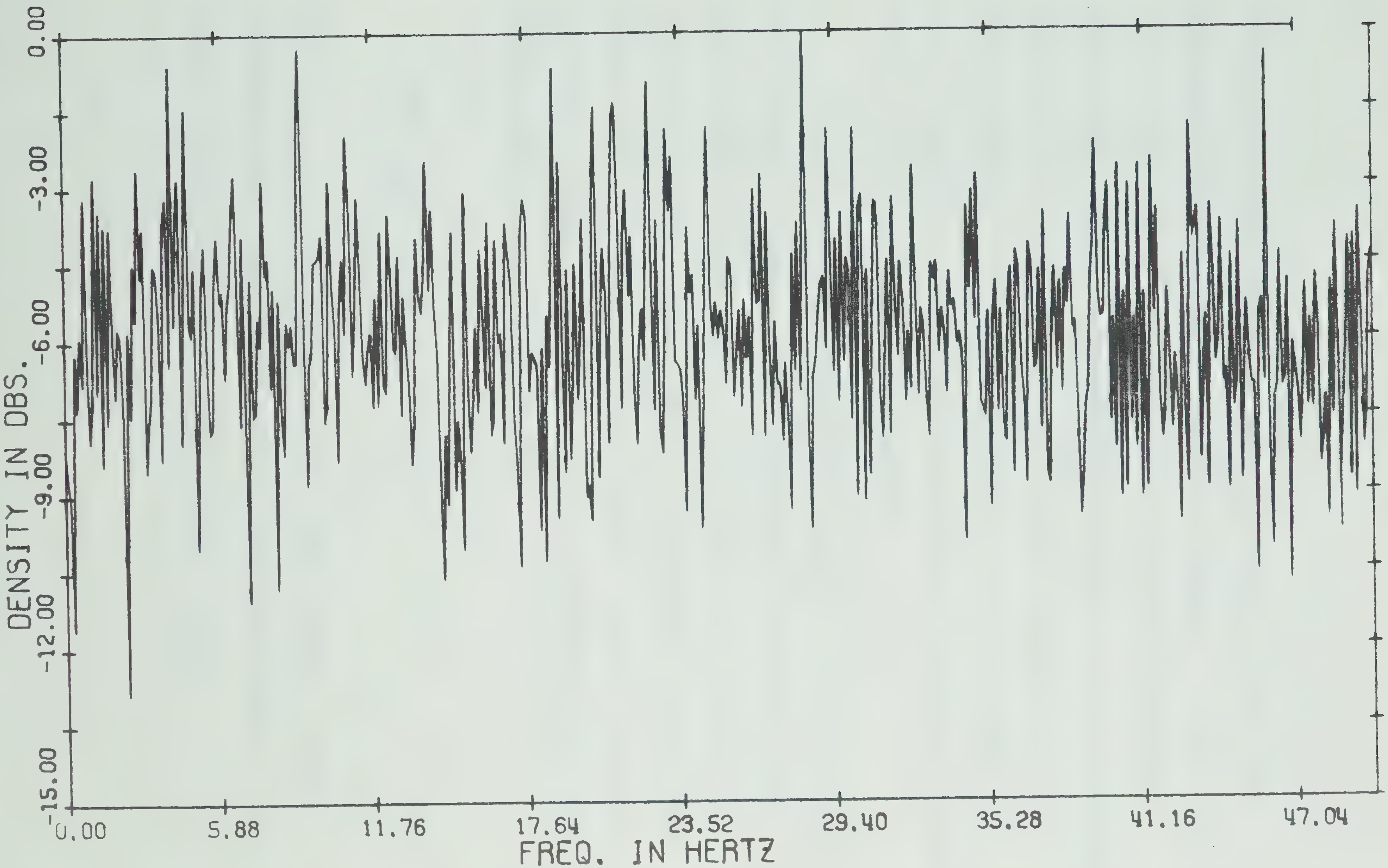

FIGURE 2.4: Power spectral estimate of the random noise sequence; normalized to its maximum value.

of the optimal regulator is to reduce the effects of the random disturbances $\underline{w}(t)$ and $\underline{v}(t)$ on vehicle performance. In the limit one could envisage a "perfect" regulator capable of completely filtering out system disturbances and thus give vehicle performance similar to that of the deterministic system of appendix one. Though obviously this "perfect" regulator is not realizable, one can still define an index of performance which gives some indication of how closely this ideal is approached.

The mean-square deviation (MSD) between the state variables of the stochastic regulator, which includes both the optimal (OSR) and the non-optimal (NOSR) regulators of figure 2.2 and figure 2.1 respectively, and the state variables of the deterministic regulator (figure A1.2) can thus be used as a gauge of stochastic regulator performance. Hence, define the estimate of the mean-square deviation (MSD) of the system variable z to be

$$MSD = \frac{1}{N_p} \sum_{k=0}^{N_p-1} \left[z(k) - z_D(k) \right]^2 \tag{2-19}$$

where $z_D(k)$, $z(k)$ are the deterministic model response and the stochastic model response at time $k\Delta T$, respectively, and N_p is the number of sample points used in the estimate. Since

$$z(t) = z_D(t) + z_S(t)$$

where $\{z_S(t)\}$ is the zero mean process which is the resultant stochastic effect of system disturbances on the variable z(t), the MSD of equation (2-19) then becomes

$$MSD = \frac{1}{N_p} \sum_{k=0}^{N_p-1} \left[z_S(t) \right]^2 \tag{2-20}$$

For large N_p, performance measure (2-19) thus indicates the variance of the disturbance term affecting z.

Note that definition (2-19) permits computation of the MSD of the disturbance in the state variable x without regard for the deterministic response of the system. This is quite useful in that it can thus be calculated while concurrently observing the deterministic and stochastic response of the system

In all simulation runs to follow, the estimate of the mean-square deviation was computed using 501 samples of the variable z(t) from time t=0 to time $t_f = k \Delta T$ where k=500.

2.6.3 Simulation results

In this subsection the OSR (optimal stochastic regulator) of figure 2.2 will be simulated for the two cases where the filter does and does not have exact knowledge of system disturbances. Its performance will then (for equivalent operating conditions) be compared with the NOSR (non-optimal stochastic regulator) of figure 2.1.

In all simulations the vehicle queue initial conditions are such that $\delta y_1=\delta y_2=\delta y_3=0$, $\delta w_1=-4.2$, and $\delta w_2=2.1$.[18] The corresponding filter states ($\hat{x}_1$, $\hat{x}_3$, $\hat{x}_5$, and $\hat{x}_2$, $\hat{x}_4$, respectively) are given identical conditions at time $t=0$.[19]

Whenever a simulation run incorporates the Kalman filter, the filter invariably assumes that the noise autocovariance matrices W and V are both identically equal to the identity matrix.

a. NON-OPTIMAL STOCHASTIC REGULATOR (NOSR) SYSTEM - Table 2-3 gives the mean-square deviations of various system variables for several values of noise variance. The observed response of the position and velocity errors are shown in figure 2-5 and figure 2-6, respectively, while the controller input signal (here called "measured state") for state variable δw_1, and the control signal for the second vehicle, δF_2, are shown in figures 2-7 and 2-8, respectively (plant and measurement noise both have a variance of one).[20]

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[18]Note that these are the same initial conditions used to obtain the deterministic regulator response shown in appendix one. The non-zero initial conditions allow concurrent observation of the deterministic and stochastic components of the vehicle queue response, and also allow easy comparison with the deterministic model response.

[19]The filter initial conditions are set identically equal to the plant's in order to avoid the tracking error that would otherwise result at the starting time.

[20]The superimposed smooth response, on these and subsequent graphs, is that of the deterministic noiseless system of appendix one. It is provided for purposes of comparison.

The indicated results clearly point out the serious consequences for vehicle performance that system disturbances can have. What is even more striking however, is their effect on the required corrective forces needed to maintain the steady-state condition. From table 2-3, for example, it is seen that (for a noise variance of one) in order to maintain the mean-square velocity deviation of the first vehicle at 0.0085 the corrective force, δF_1, mean square deviation required is 9.5185.

b. OPTIMAL STOCHASTIC REGULATOR (OSR) SYSTEM - Table 2-4 gives the mean-square deviations of various system variables for several values of noise variance. In all cases, the designed Kalman filter assumes (for both measurement and plant noise) a variance of one.

Figure 2-9 to figure 2-13 gives the graphical responses of several variables when the filter has exact knowledge of the system noise (i.e. the actual noise variance is one). Figure 2-14 to figure 2-18 show the graphical responses of those same variables when the filter has an inaccurate description of system noise (i.e. actual noise variance is 9.0 but the filter assumes it to be 1.0).

The remarkably improved response that the Kalman filter makes possible is quite obvious. As a comparison with part a (no Kalman filter), to maintain the mean-square velocity deviations of the first vehicle to 0.0024 requires a corrective force mean-square deviation, δF_1, of 0.0070 (for a system noise variance of one).

TABLE 2-3

Mean-square deviations for the NOSR system*

σ_N^2	MEAN-SQUARE DEVIATIONS		
	$\underline{x}$	$\underline{y}$	$\underline{u}$
1.0	0.0085 0.0082 0.0067 0.0038 0.0033	1.0743 0.9809 1.0793 1.0994 0.9709	9.5184 11.5912 9.8712
4.0	0.0304 0.0331 0.0268 0.0155 0.0132	4.2977 3.9240 4.3174 4.3978 3.8839	38.0742 46.3664 39.4850
9.0	0.0766 0.0744 0.0604 0.0350 0.0297	9.6697 8.8290 9.7140 9.8950 8.7389	85.6689 104.3263 88.8431
16.0	0.1361 0.1324 0.1074 0.0622 0.0529	17.1902 15.6956 17.2690 17.5908 15.5353	152.2956 185.4603 157.9402

* The values are given in the following order (subsequent tables included)

$\underline{x}$=state variables = $[\delta y_1 \delta w_1 \delta y_2 \delta w_2 \delta y_3]'$

$\underline{y}$=measured states = $[y_1 y_2 y_3 y_4 y_5]'$

$\underline{u}$=corrective forces=$[\delta F_1 \ \delta F_2 \ \delta F_3]'$

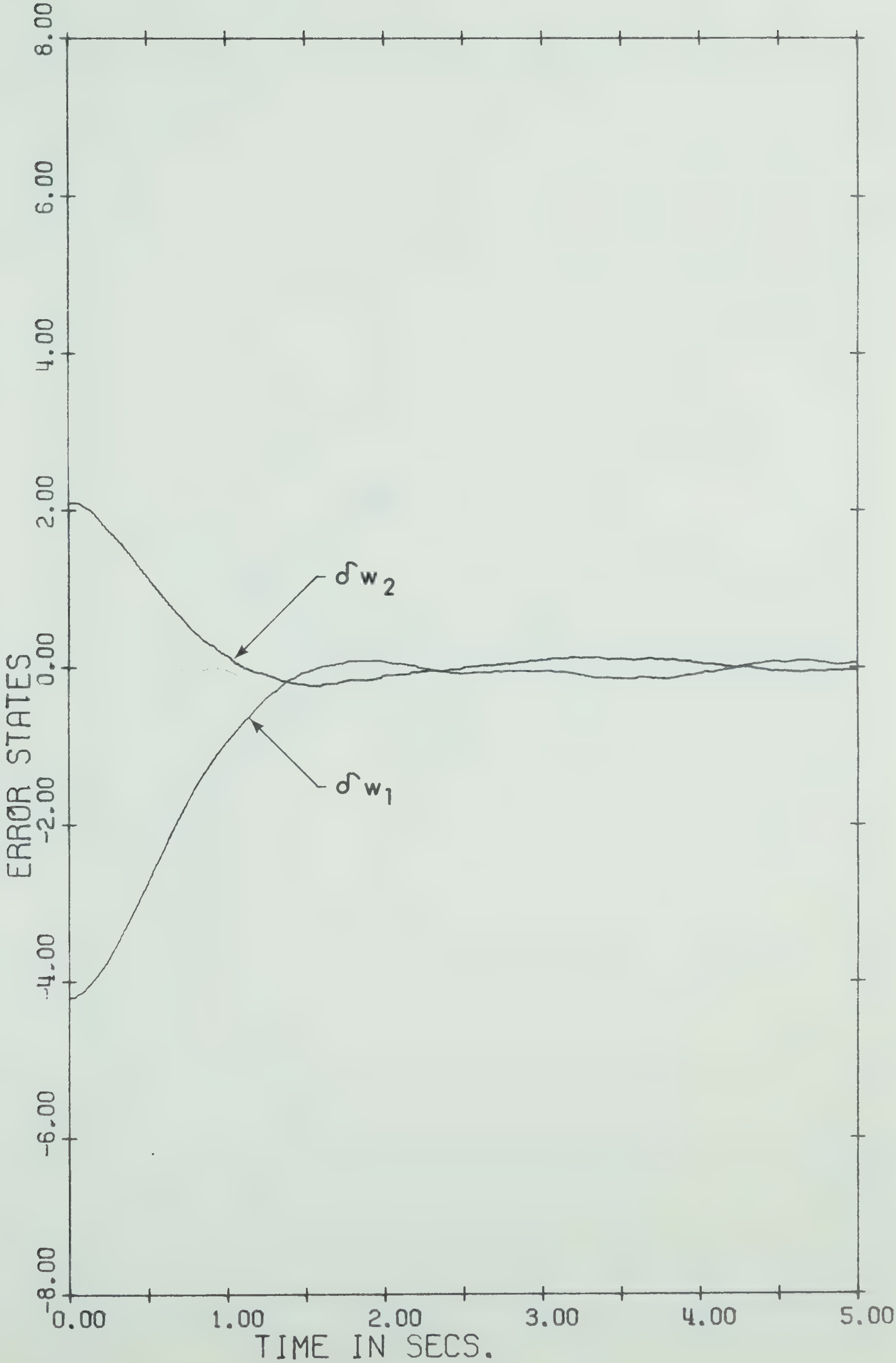


FIGURE 2.5: Vehicle position deviations with the NOSR: $\sigma_N^2=1$.

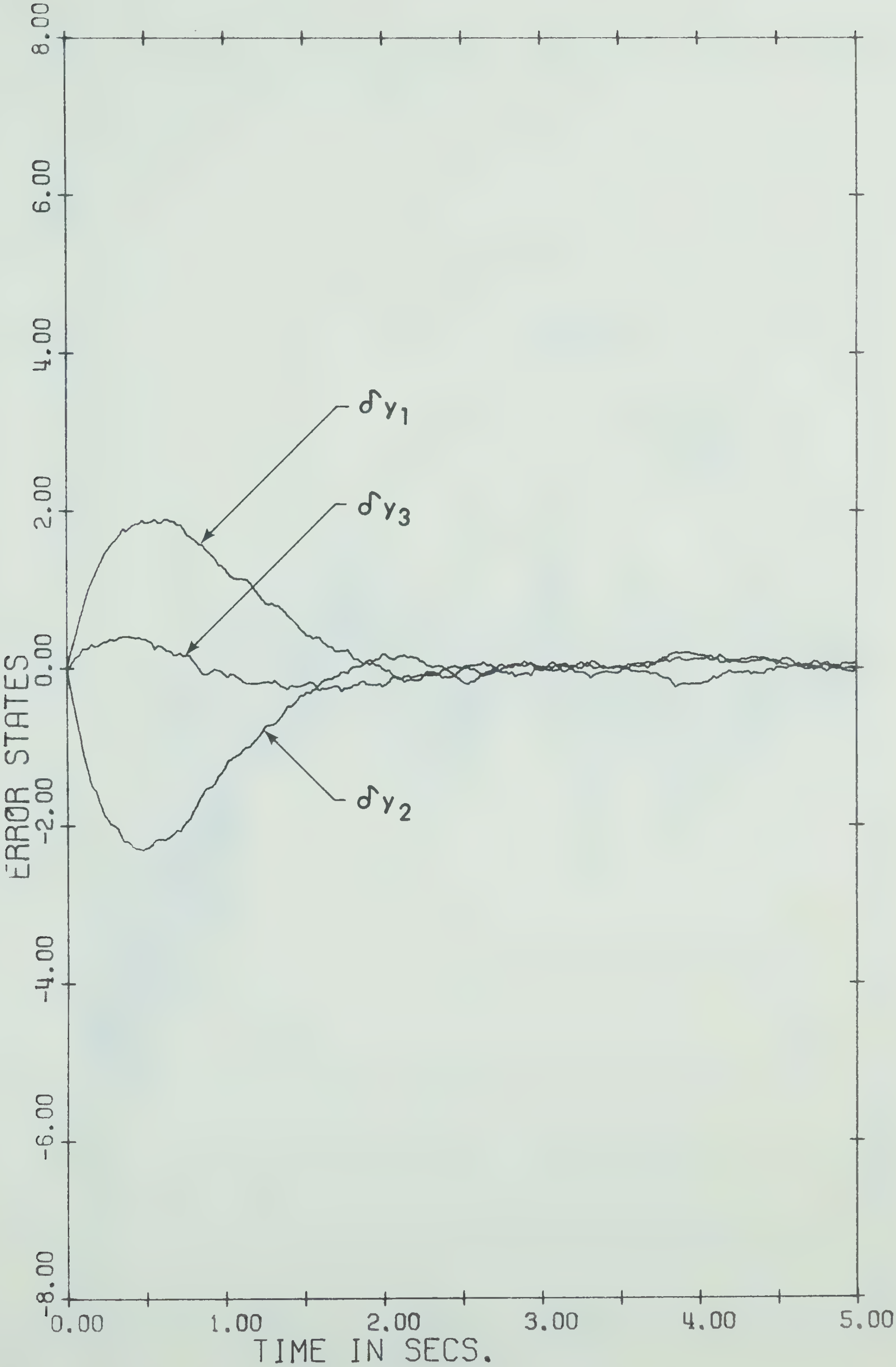


FIGURE 2.6: Vehicle velocity deviations with the NOSR: $\sigma_N^2 = 1$.

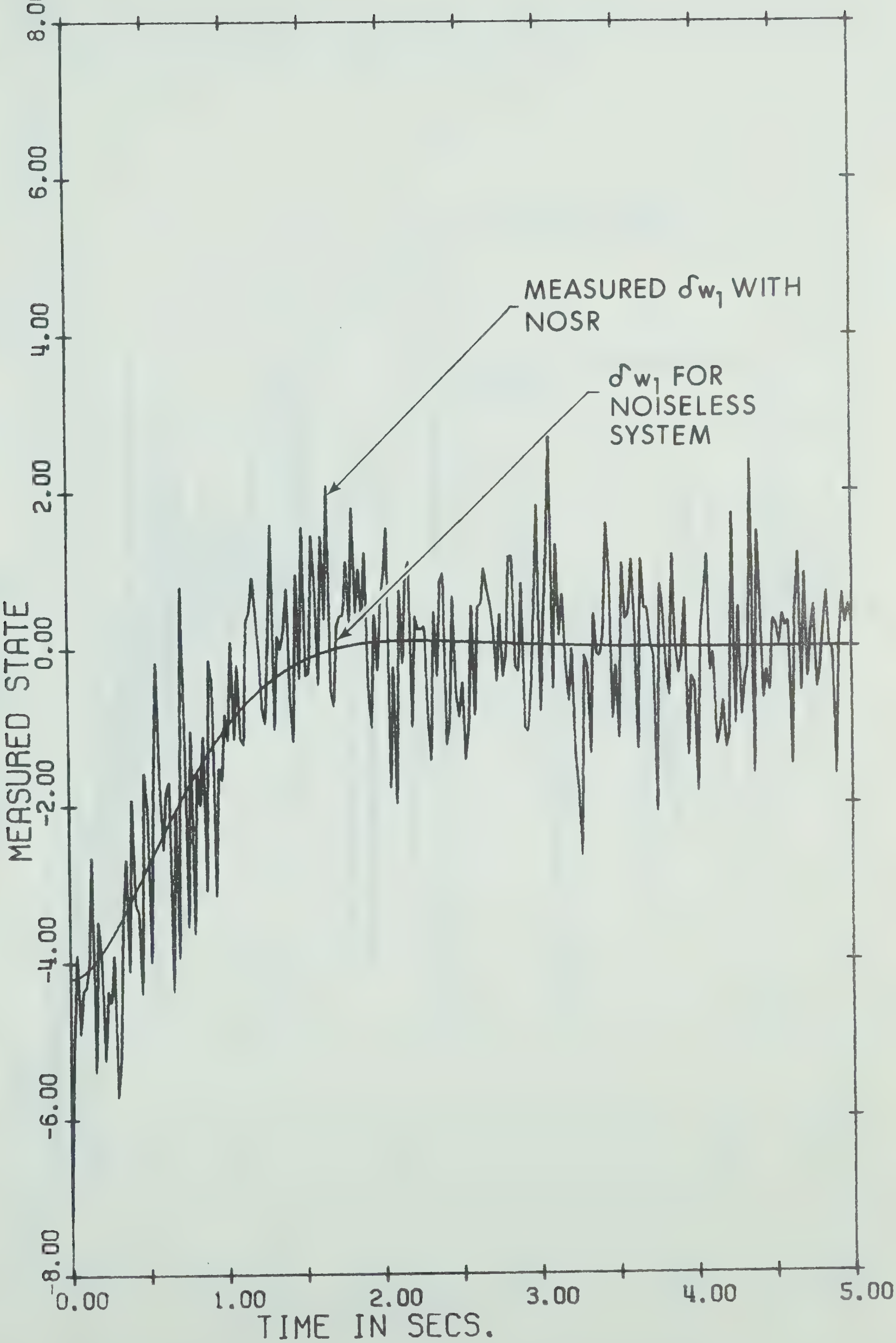


FIGURE 2.7: Measured position deviation δw_1 with the NOSR: $\sigma_N^2 = 1$.

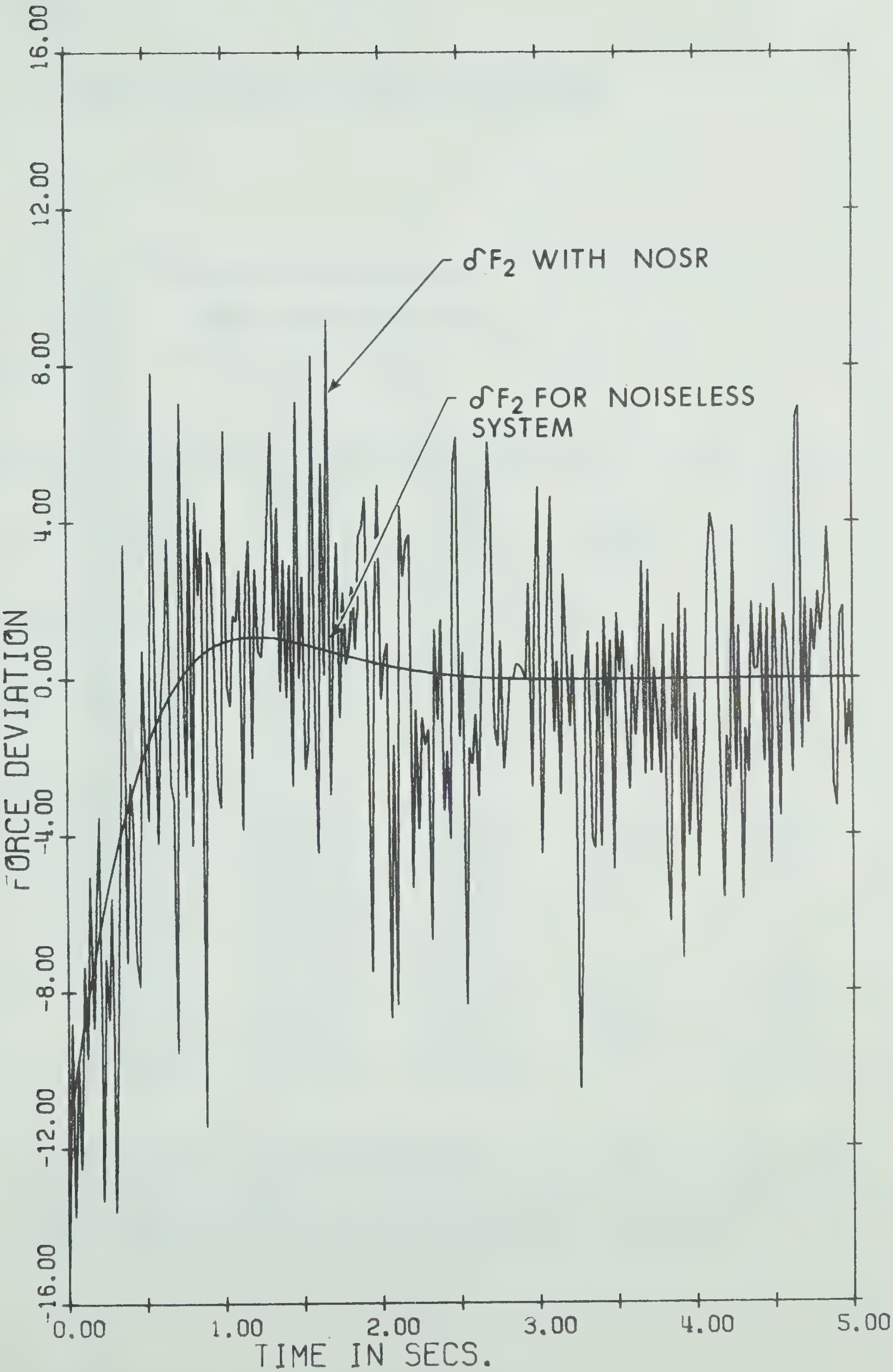


FIGURE 2.8: Control for second vehicle (δF_2) with the NOSR: $\sigma_N^2=1$.

TABLE 2-4

Mean-square deviations for the OSR system

σ_N^2	MEAN-SQUARE DEVIATIONS				
	$\underline{x}$	$\underline{y}$	$\hat{\underline{x}}$	$\underline{u}$	$\hat{\underline{x}}^{\dagger}$
1.0	0.0024 0.0037 0.0011 0.0013 0.0014	0.9903 0.9505 1.0215 1.0526 0.9673	0.0005 0.0017 0.0005 0.0010 0.0003	0.0070 0.0080 0.0037	0.0016 0.0024 0.0022 0.0022 0.0014
4.0	0.0095 0.0151 0.0047 0.0052 0.0056	3.9615 3.8022 4.0862 4.2105 3.8693	0.0020 0.0069 0.0020 0.0043 0.0013	0.0281 0.0323 0.0151	0.0067 0.0099 0.0091 0.0091 0.0059
9.0	0.0215 0.0341 0.0107 0.0117 0.0127	8.9134 8.5551 9.1938 9.4736 8.7061	0.0044 0.0156 0.0045 0.0097 0.0030	0.0633 0.0727 0.0340	0.0151 0.0222 0.0206 0.0204 0.0133
16.0	0.0383 0.0607 0.0190 0.0208 0.0226	15.8455 15.2083 16.3441 16.8417 15.4769	0.0079 0.0278 0.0081 0.0173 0.0054	0.1126 0.1293 0.0604	0.0269 0.0396 0.0367 0.0364 0.0236

† These are given for $[\delta y_1 \delta w_1 \delta y_2 \delta w_2 \delta y_3]'$. The values are obtained using equation (2-35) and replacing the deterministic model response, $Z_D(k)$, with the actual state response, $Z(k)$. It thus gives the MSD between the estimated and the actual system states.

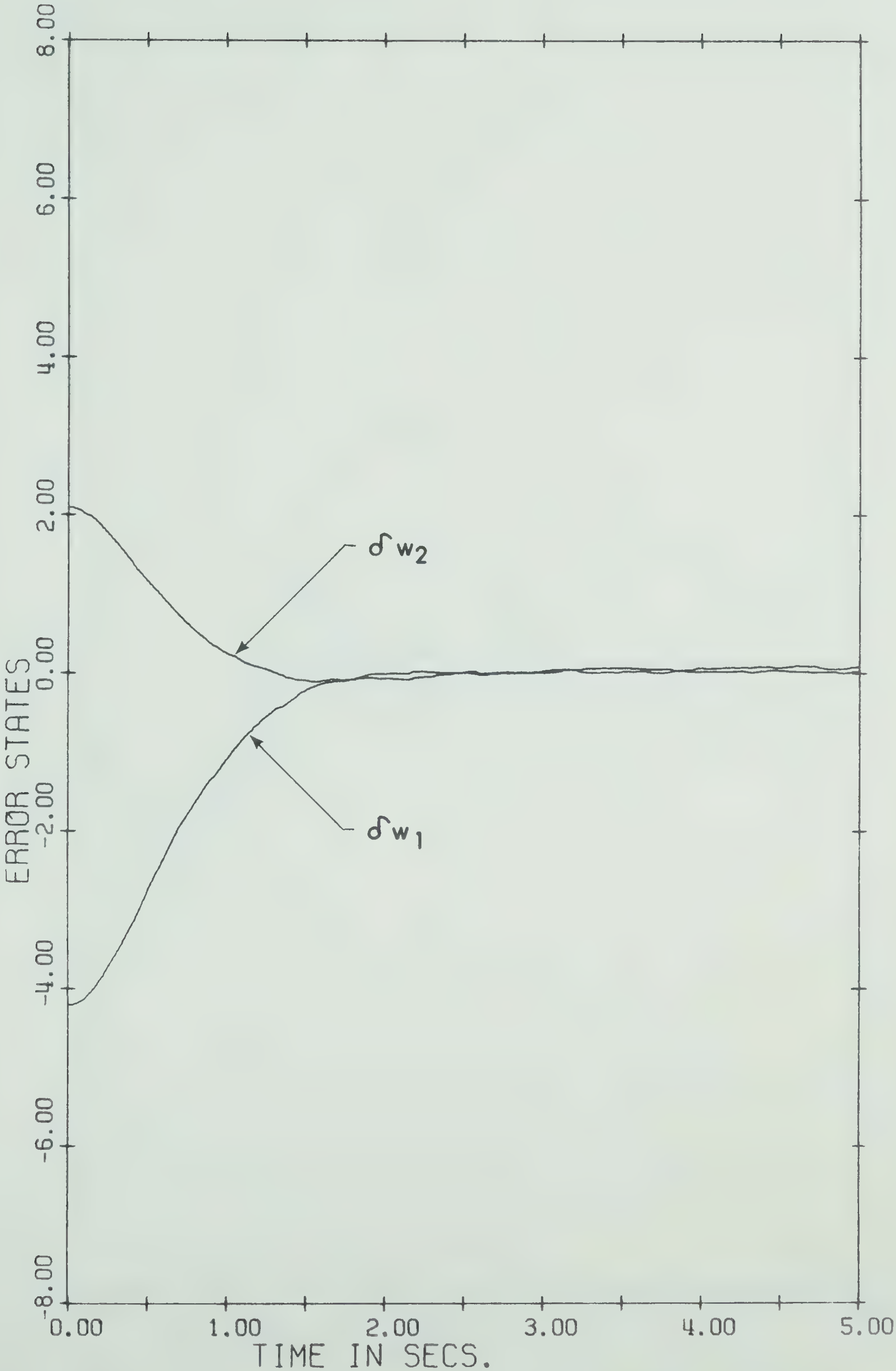


FIGURE 2.9: Vehicle position deviations with the OSR: $\sigma_N^2=1$.

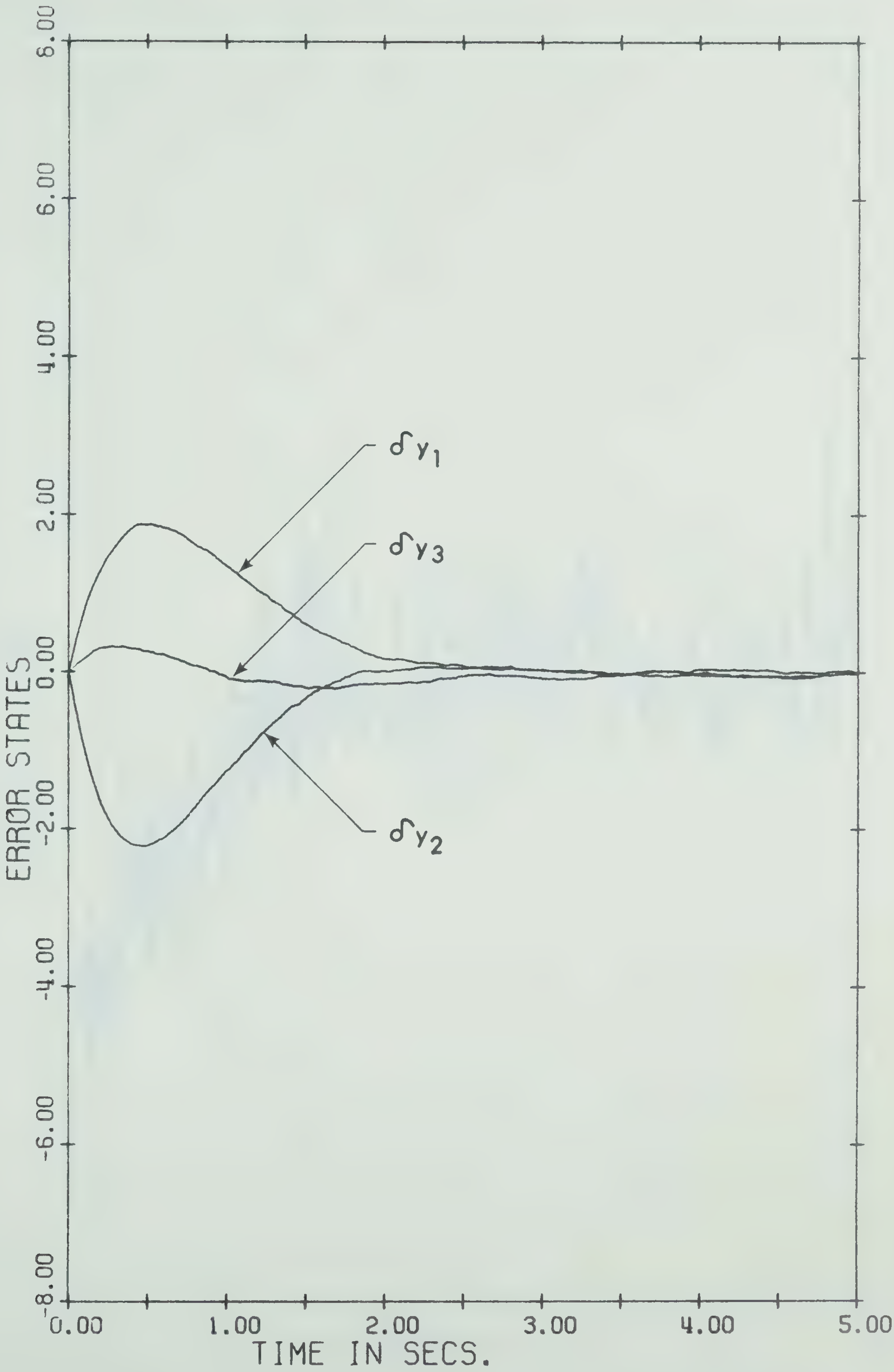


FIGURE 2.10: Vehicle velocity deviations with the OSR: $\sigma_N^2=1$.

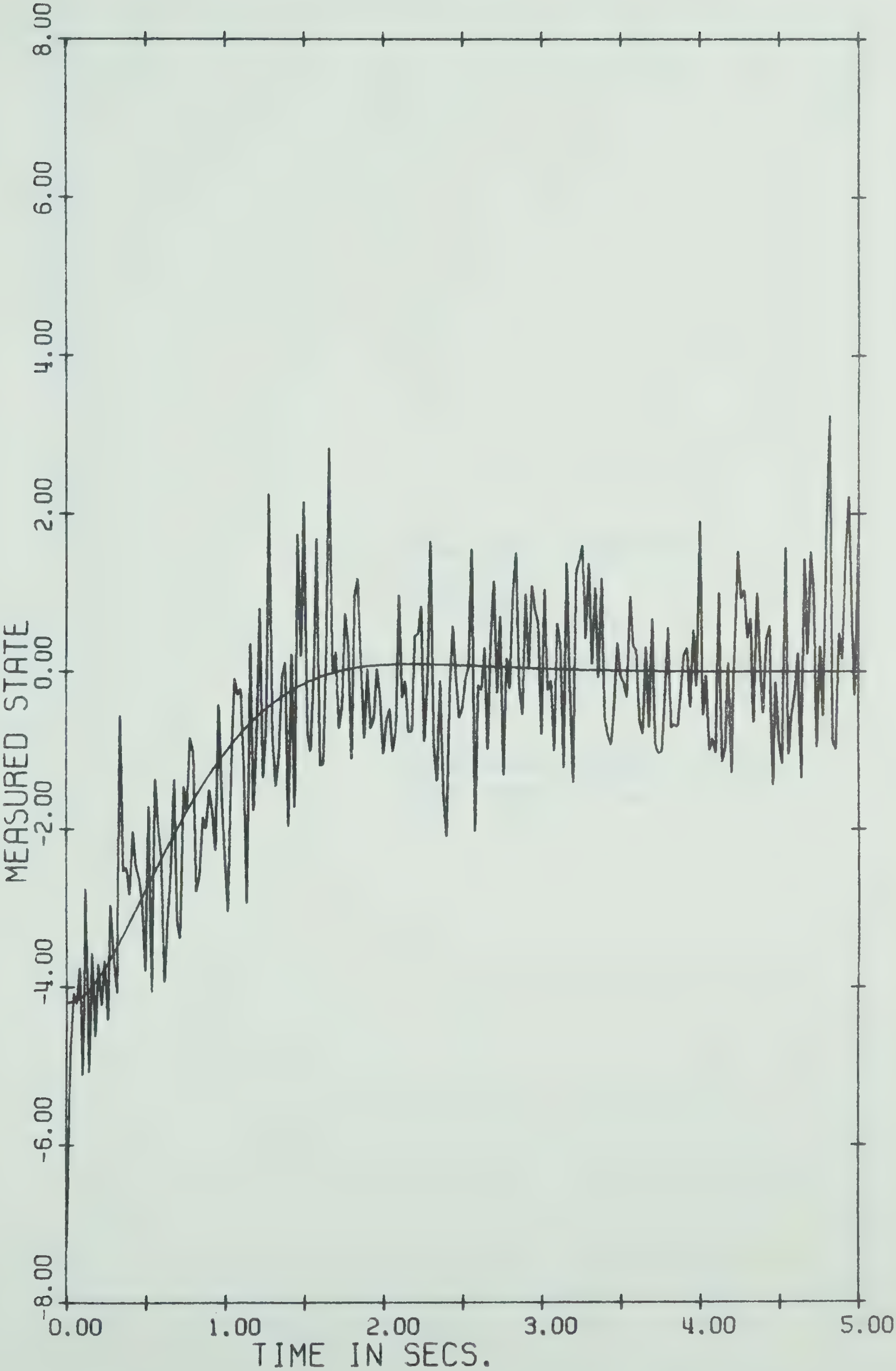


FIGURE 2.11: Measured position deviation δw_1 with the OSR: $\sigma_N^2=1$.

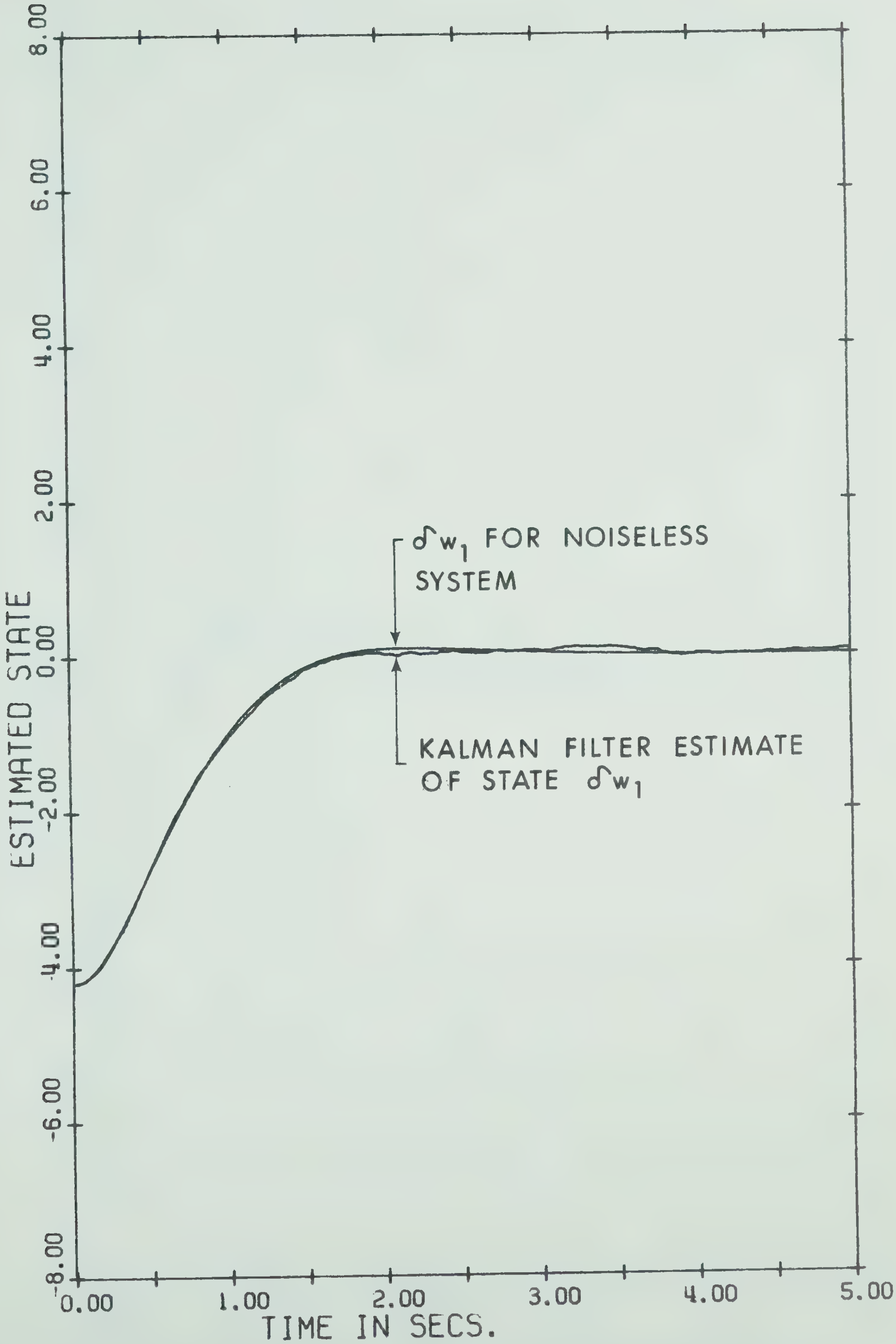


FIGURE 2.12: Kalman filter estimate of the state $\delta w_1 : \sigma_N^2 = 1$.

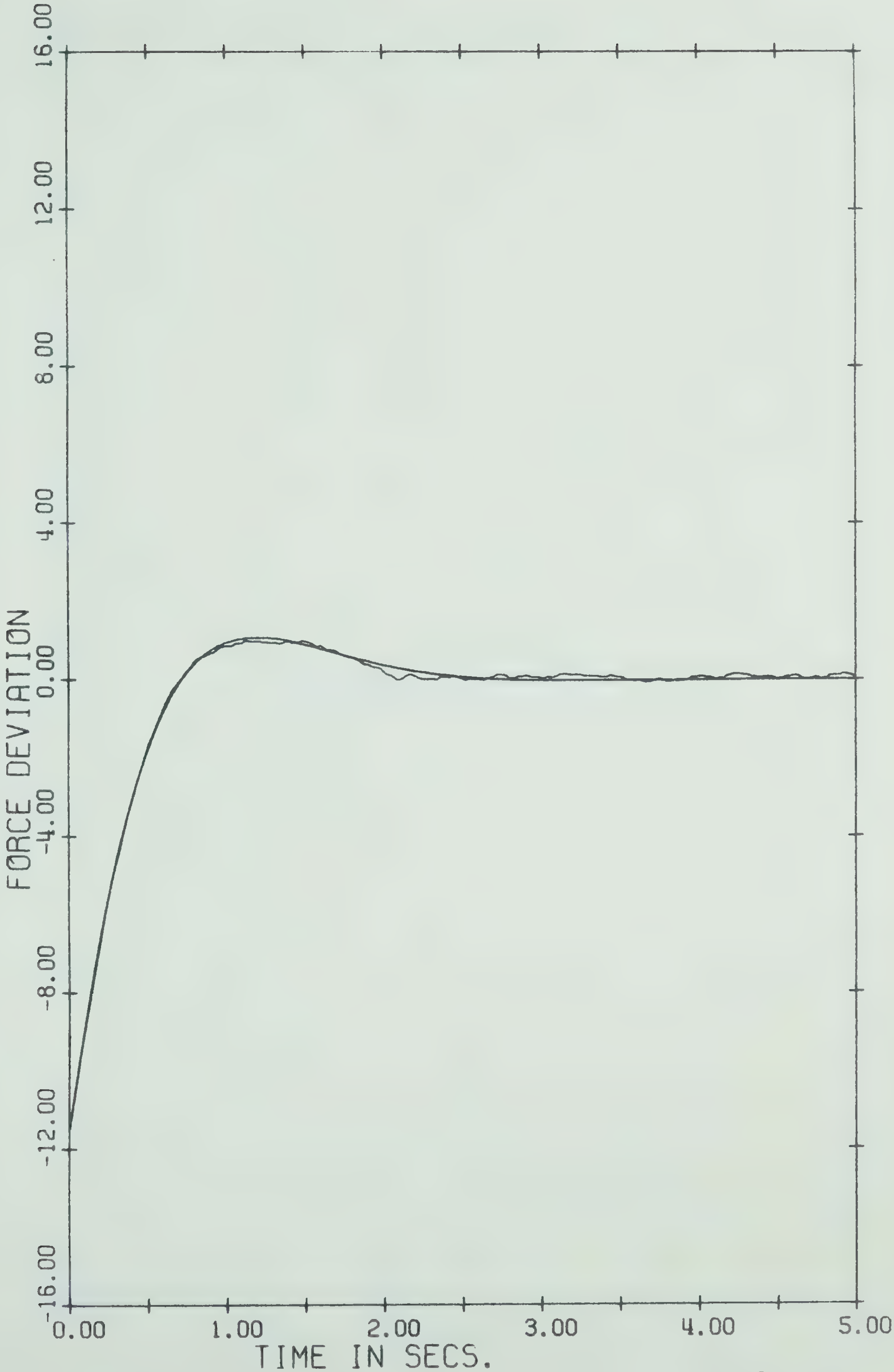


FIGURE 2.13: Control for second vehicle (δF_2) with the OSR: $\sigma_N^2=1$.

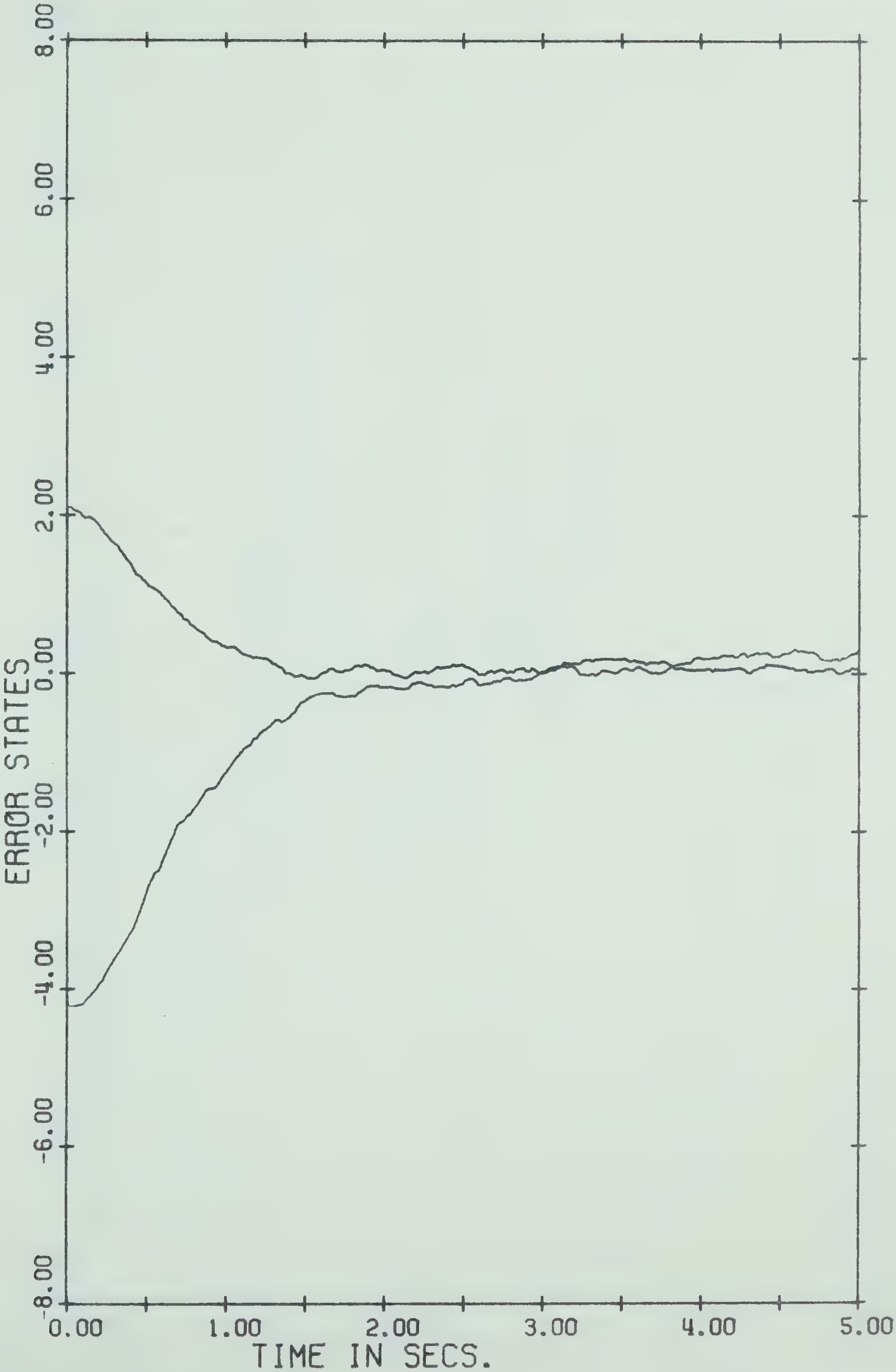


FIGURE 2.14: Vehicle position deviations with the OSR: $\sigma_N^2=9$.

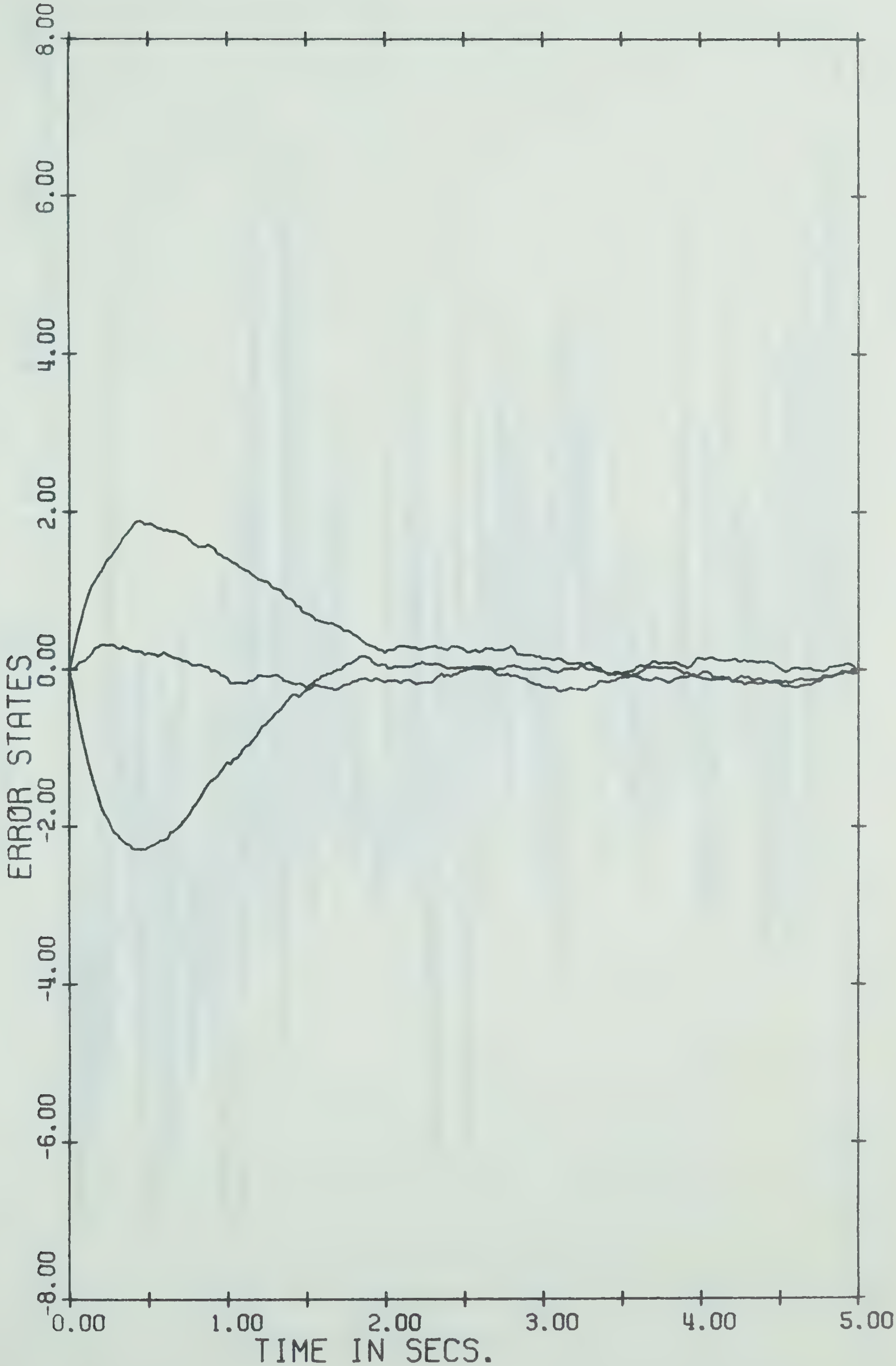


FIGURE 2.15: Vehicle velocity deviations with the OSR: $\sigma_N^2=9$.

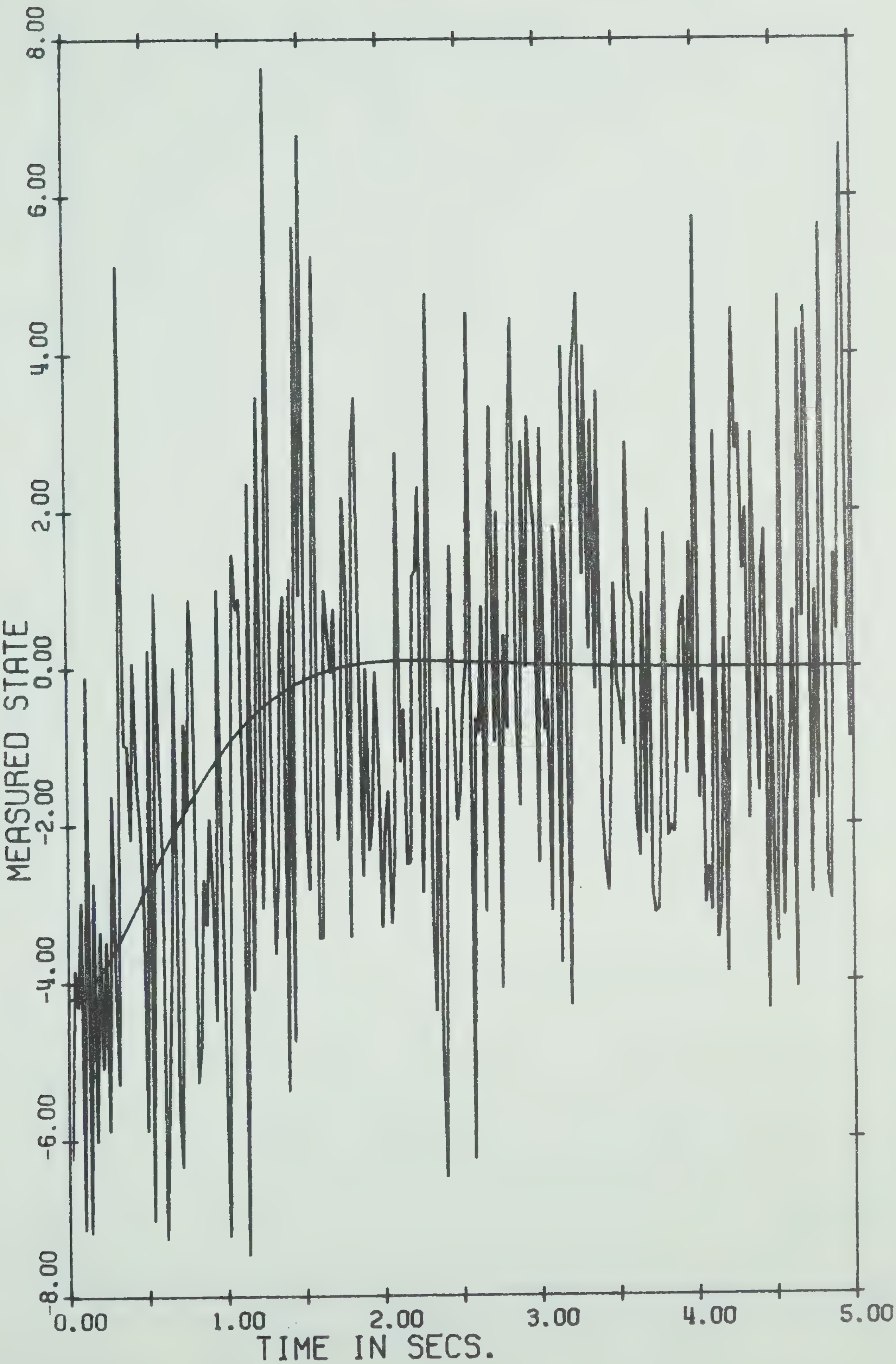


FIGURE 2.16: Measured position deviation δw_1 with the OSR: $\sigma_N^2=9$.

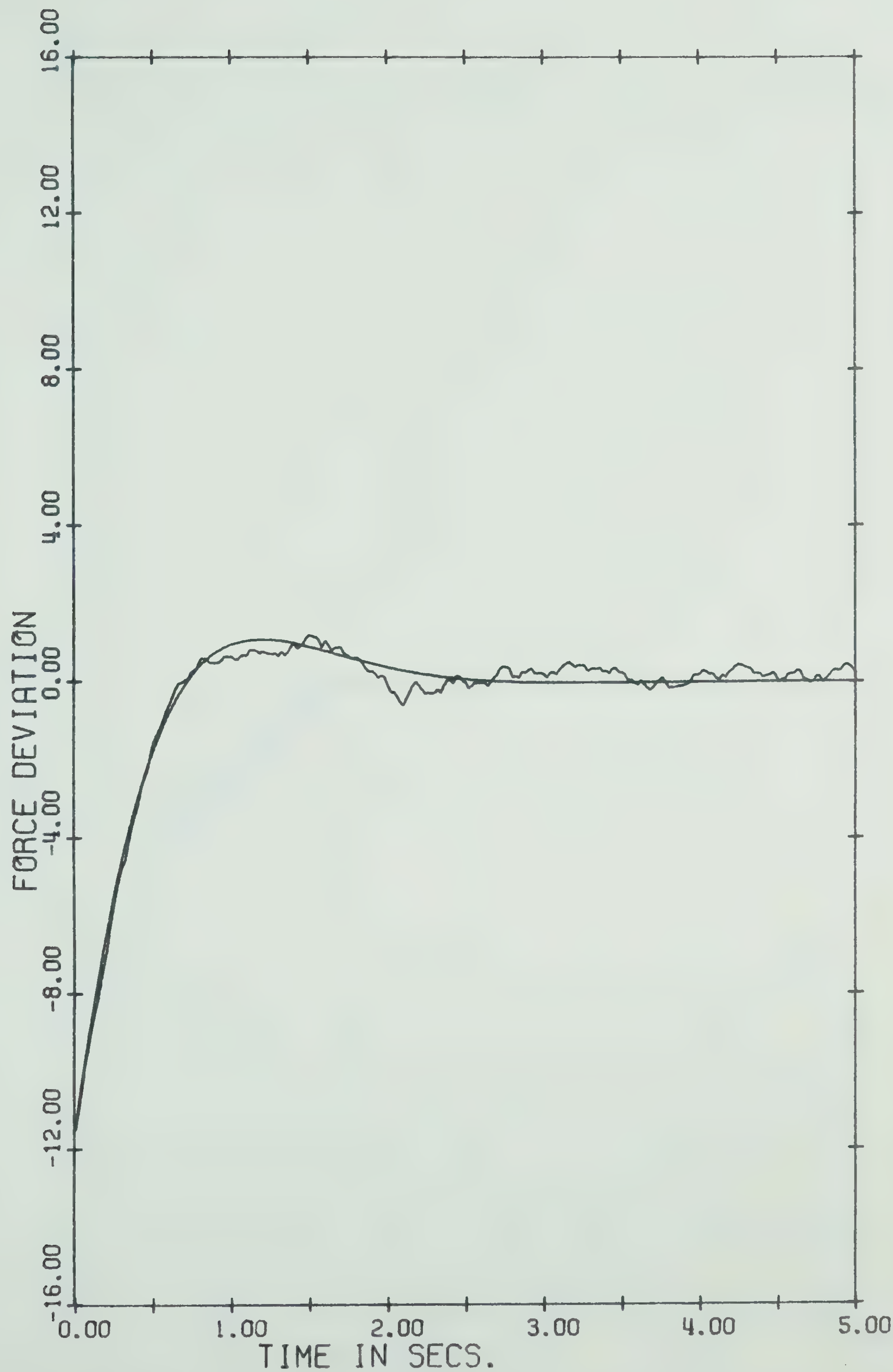


FIGURE 2.17: Kalman filter estimate of the state $\sigma_N^2=9$.

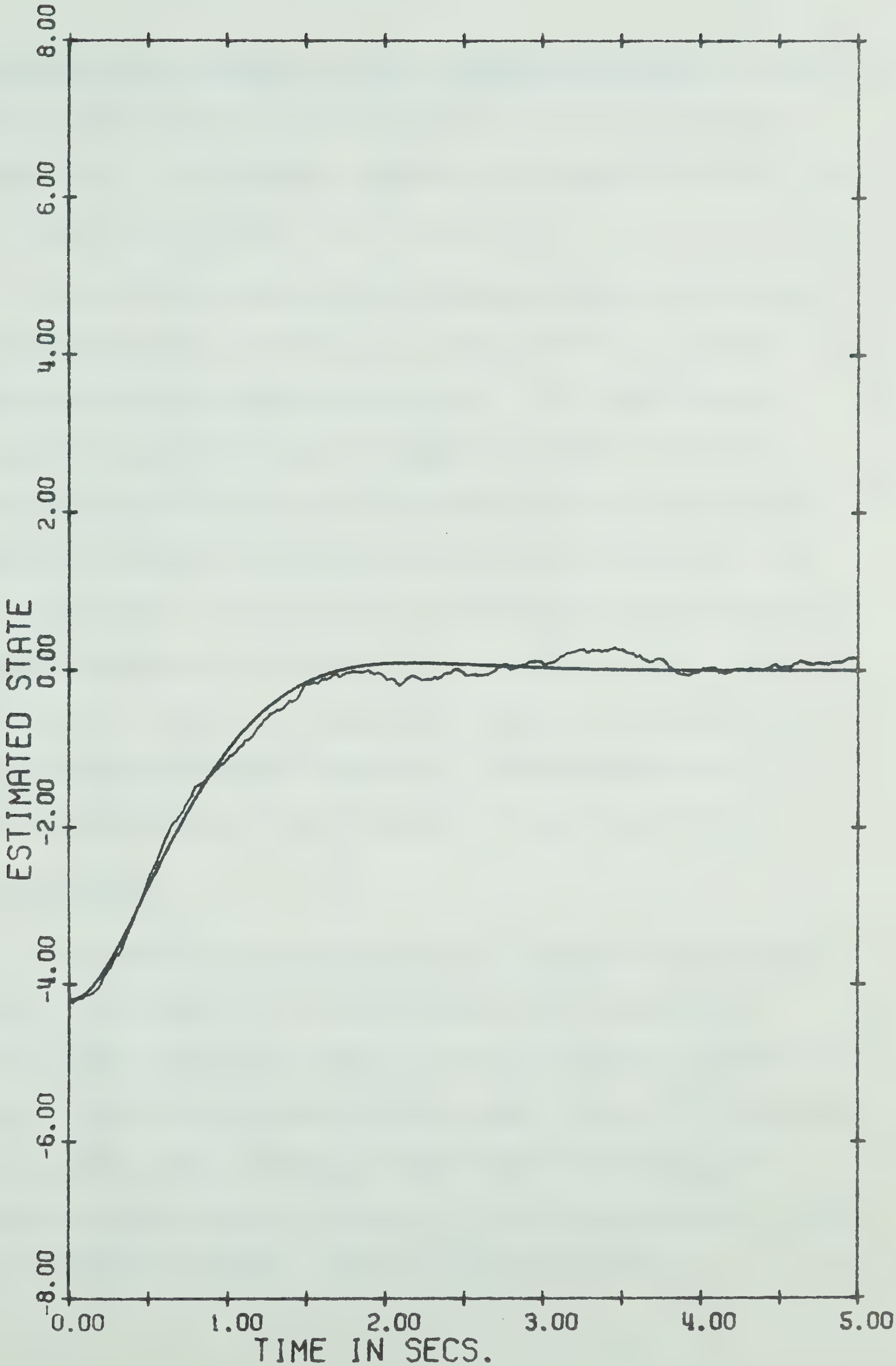


FIGURE 2.18: Control for second vehicle (δF_2) with the OSR: $\sigma_N^2=9$.

c. <u>SYSTEM WITH DISTURBANCES HAVING A ~~UNIFORM~~ DISTRIBUTION</u>- The equations for the conditional mean estimate hinge on the linear-Gaussian assumptions. If the Gaussian assumption is relaxed, then (2-8, 2-9) do not give the conditional mean estimate [42].

Table 2-5 and table 2-6 give the mean-square deviations for the NOSR and the OSR, respectively, in the presence of system disturbances having a Normal distribution. Unlike the previous regulators subjected to Gaussian random disturbances, it is now noted that the ability of the OSR to regulate the position of the controlled vehicles is actually less than that of the NOSR. This is a direct result of the inability of the Kalman filter in the OSR system to accurately estimate the actual vehicle states. Since cost function (2-11) forces the regulators to take corrective action in the presence of positional errors only, the poorer position regulation achieved with the OSR system is readily explainable.

2.7 Conclusions

As a sequel to the qualitative work reported by Anderson and Powner [1], a comparison of table 2-3 with table 2-4 clearly indicates the immense improvement in vehicle response obtainable with the OSR; over that obtainable with the NOSR. Figure 2.19 graphically compares (for noise variances of from 1 to 16) the mean-square deviation of the velocity variable, δy_1, and of the corrective force, δF_1, of the first vehicle, when the NOSR and the OSR are

TABLE 2-5

Mean-square deviations for the NOSR system : disturbances have a Normal distribution.

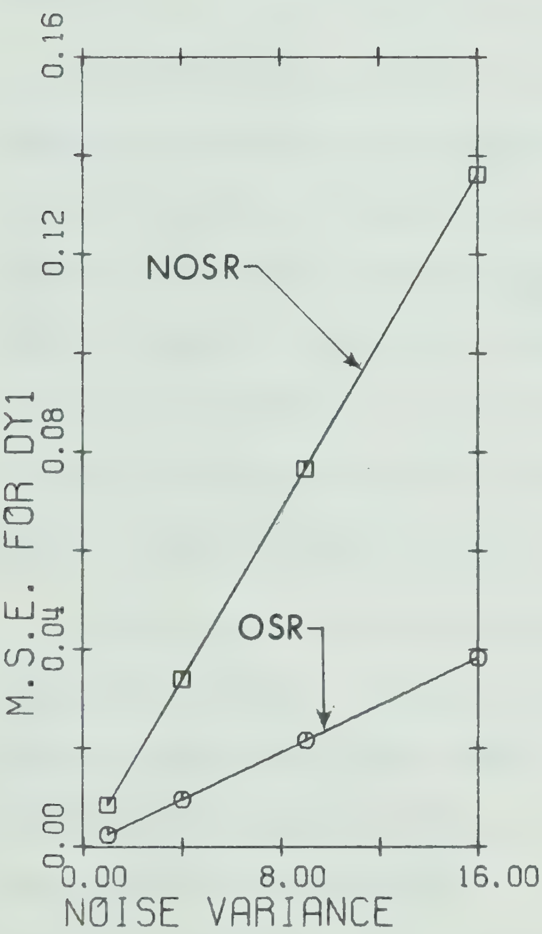
$\underline{x}$	$\underline{y}$	$\underline{u}$
0.0010	0.3587	3.0334
0.0003	0.3277	3.6812
0.0015	0.3119	3.4407
0.0010	0.3567	3.4407
0.0019	0.3276	

TABLE 2-6

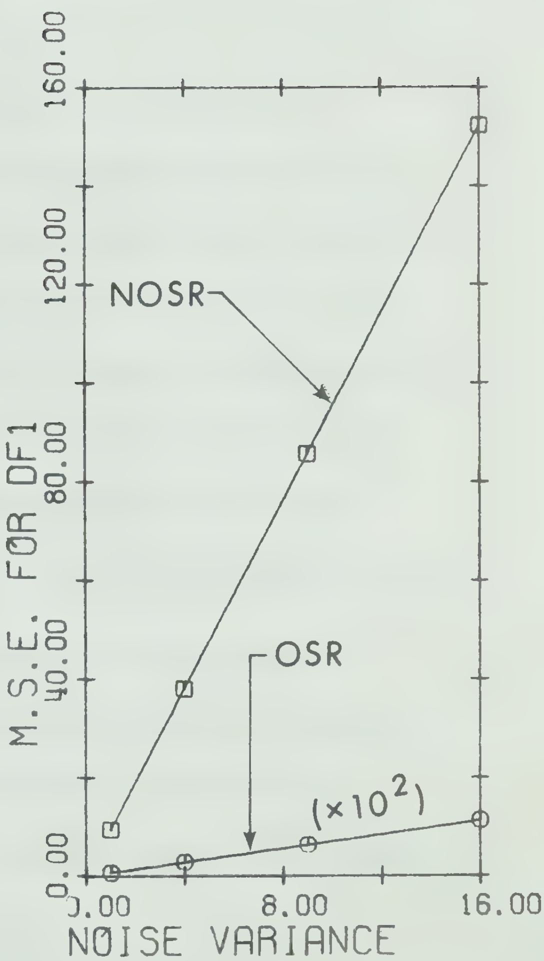
Mean-square deviations for the OSR system : disturbances have a Normal distribution.

$\underline{x}$	$\underline{y}$	$\underline{\hat{x}}$	$\underline{u}$	$\underline{\hat{x}}$ †
0.0004	0.3363	0.0002		0.0007
0.0032	0.3062	0.0004	0.0018	0.0020
0.0007	0.3338	0.0004	0.0035	0.0002
0.0011	0.3410	0.0018	0.0045	0.0005
0.0018	0.3314	0.0008		0.0004

† Comments in the footnote of Table 2-4 apply



(a)



(b)

FIGURE 2.19: (a) MSD of velocity error variable of first vehicle.
(b) MSD of corrective force on first vehicle.

employed.[21] It is seen that, as the noise variance increases, the OSR system gives a better response than the NOSR system even though, except for the case of unity noise variance, the filter in the OSR system does not have exact knowledge of system disturbances.

By far the most obvious effect of the filter is its phenomenal reduction in the magnitude of the required corrective forces to maintain a relatively stable steady-state queue condition. Figure 2-19b compares the mean-square deviation of the corrective force δF_1 applied to the first vehicle of the queue when the filter is and is not included in the system. The attendant benefits that this has for passenger comfort and system operating costs have been discussed in chapter one and will not be considered further here.

Given the vehicle queue with both plant and measurement disturbances (having a Gaussian distribution) the optimal stochastic regulator system thus consistently permits better vehicle regulation even though the filter in the OSR may not have exact knowledge of system disturbances.

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[21] It should be apparent from the previous discussion that when the OSR is used, the filter estimate is optimal only for the case where the actual noise variance is 1. Had the filter been redesigned each time the noise variance changed, the MSD of the system states with the Kalman filter would have shown a rather horizontal relation.

CHAPTER THREE

THE VEHICLE SYSTEM WITH PLANT NOISE AND FEEDBACK TIME DELAY

ABSTRACT

An optimal closed loop control scheme incorporating a least mean-squared predictor is first developed for a three-vehicle system with plant noise and feedback time delay. Optimal regulators are designed for two different performance criteria. These resulting optimal closed loop systems are then simulated with different amounts of time delay in the feedback loop. The effects of an incorrectly assumed feedback delay on performance is also examined.

3.1 The least-mean squared predictor and feedback control

Consider the stochastic steady-state vehicle queue model described in sub section 2.1 and suppose that now (as discussed in chapter one) there is a time delay between the acquisition of the state measurement $\underline{x}(t)$ and the generation of the required control action $\underline{u}(t)$. The state-output equations of the vehicle queue can thus be taken as

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) + \underline{w}(t) \tag{3-1a}$$

$$\underline{y}(t) = \underline{x}(t-\tau) \tag{3-1b}$$

where τ is the time delay. Equation (3-1b) assumes a completely undistorted controller input signal $\underline{y}(t)$ of the delayed state $\underline{x}(t)$. The admissable control input which minimizes the quadratic cost function $J_E(\underline{u})$ of equation (2-5) subject to dynamic constraints (3-1), $\underline{u}^*(t)$, is now desired.

It is a well known result that the form of the prediction process is expressed as the conditional expectation of $\underline{x}(t)$, [29, 24, 31]

$$\hat{\underline{x}}(t) = E\{\underline{x}(t)/\underline{y}(\sigma),\ \sigma \leq t\} \ . \tag{3-2}$$

Since the estimate is to be obtained in terms of the received signal (output) $\{\underline{y}(\sigma),\sigma\leq t\}$ the state $\underline{x}(t)$ should be expressed as a function of $\underline{y}(t)$. We therefore write

$$\dot{\underline{y}}(t) = \dot{\underline{x}}(t-\tau) = A\,\underline{x}(t-\tau) + B\,\underline{u}(t-\tau) + \underline{w}(t-\tau)$$

$$= A\,\underline{y}(t) + B\,\underline{u}\,(t-\tau) + \underline{w}(t-\tau). \tag{3-3}$$

If we let

$$\underline{y}(t) = \underline{y}_D(t) + \underline{y}_S(t) \tag{3-4}$$

where $\underline{y}_D(t)$, $\underline{y}_S(t)$ are the deterministic and the stochastic parts of the system output respectively, then equation (3-3) becomes

$$(\dot{\underline{y}}_D(t) + \dot{\underline{y}}_S(t)) = A(\underline{y}_D(t) + \underline{y}_S(t)) + B\,\underline{u}(t-\tau)$$

$$+ \underline{w}(t-\tau) \quad . \tag{3-5}$$

$\underline{y}_S(t)$ is a purely random, zero mean, white noise term because of the assumptions on $\underline{w}(t)$. From the linearity of the system, the state variables can then be separated to obtain

$$\dot{\underline{y}}_D(t) = A\,\underline{y}_D(t) + B\,\underline{u}\,(t-\tau) \tag{3-6}$$

$$\dot{\underline{y}}_S(t) = A\ \underline{y}_S(t) + \underline{w}(t-\tau) \quad . \tag{3-7}$$

Now, from (3-4) and (3-1b) we have

$$\underline{y}(t+\tau) = \underline{y}_D(t+\tau) + \underline{y}_S(t+\tau) = \underline{x}(t) \quad . \tag{3-8}$$

The conditional expectation of $\underline{x}(t)$ given by (3-2) can be re-written in view of (3-8) as follows

$$\hat{\underline{x}}(t) = E\ \{\underline{x}(t)/\underline{y}(\sigma),\sigma\leq t\}$$

$$= E\{\underline{y}_D(t+\tau) + \underline{y}_S(t+\tau)/\underline{y}(\sigma),\sigma\leq t\}$$

$$= \underline{y}_D(t+\tau) + E\{\underline{y}_S(t+\tau)/\underline{y}(\sigma),\sigma\leq t\} \quad . \tag{3-9}$$

The least mean-squared predictor, $\hat{\underline{x}}(t)$, now requires specification of the second term in (3-9).

The solution of the differential equation

$$\dot{\underline{y}}_S(t+\tau) = A\ \underline{y}_S(t+\tau) + \underline{w}(t) \tag{3-10}$$

is given by

$$\underline{y}_S(t+\tau) = \ell^{A(t+\tau-t_o)}\,\underline{y}_S(t_o) + \int_{t_o}^{t+\tau} \ell^{A(t+\tau-\sigma)}\,\underline{w}(\sigma-\tau)\,d\sigma$$

$$= \ell^{A(t+\tau-t_o)}\,\underline{y}_S(t_o) + \int_{t_o}^{t} \ell^{A(t+\tau-\sigma)}\,\underline{w}(\sigma-\tau)d\sigma + \int_{t}^{t+\tau} \ell^{A(t+\tau-\sigma)}\underline{w}(\sigma-\tau)d\sigma \qquad (3\text{-}11)$$

or

$$\underline{y}_S(t+\tau) = \hat{\underline{y}}_S(t+\tau) + \int_{t}^{t+\tau} \ell^{A(t+\tau-\sigma)}\underline{w}(\sigma-\tau)d\sigma$$

where

$$\hat{\underline{y}}_S(t+\tau) = \ell^{A(t+\tau-t_o)}\,\underline{y}_S(t_o) + \int_{t_o}^{t} \ell^{A(t+\tau-\sigma)}\,\underline{w}(\sigma-\tau)d\sigma \quad .$$

$\hat{\underline{y}}_S(t+\tau)$ is the estimate of $\underline{y}_S(t+\tau)$ based on the observation of white noise up to time t.

$$\hat{\underline{y}}_S(t+\tau) = \ell^{A\tau}\,\{\ell^{A(t-t_o)}\,\underline{y}_S(t_o) + \int_{t_o}^{t} \ell^{A(t-\sigma)}\,\underline{w}(\sigma-\tau)d\sigma\} \qquad (3\text{-}12)$$

From

$$\hat{\underline{y}}_S(t+\tau) = \ell^{A\tau}\underline{y}_S(t) \quad ,$$

given $\{\underline{y}(\sigma), \sigma \leq t\}$, $\hat{\underline{y}}_S(t+\tau)$ is determined.

Thus

$$E\{\underline{y}_S(t+\tau)/\underline{y}(\sigma), \sigma \leq t\} = E\{\underline{y}_S(t+\tau)/\underline{y}_S(\sigma), \sigma \leq t\}$$

$$= \ell^{A\tau}\ \underline{y}_S(t).^{22} \tag{3-13}$$

The least mean-squared predictor is then given by

$$\hat{\underline{x}}(t) = \underline{y}_D(t+\tau) + \ell^{A\tau}\ \underline{y}_S(t) \tag{3-14}$$

From chapter two, the optimal feedback control is

$$\underline{u}^*(t) = \phi(t, \hat{\underline{x}}(t)) \tag{2-6}$$

---

22 Kleinman [23] obtains a similar result.

which, for reasons previously discussed, becomes

$$\underline{u}^*(t) = -R^{-1}B' \hat{K} \hat{\underline{x}}(t) = - L^* \hat{\underline{x}}(t) \tag{3-15}$$

Figure 3.1 shows the optimal stochastic feedback system.

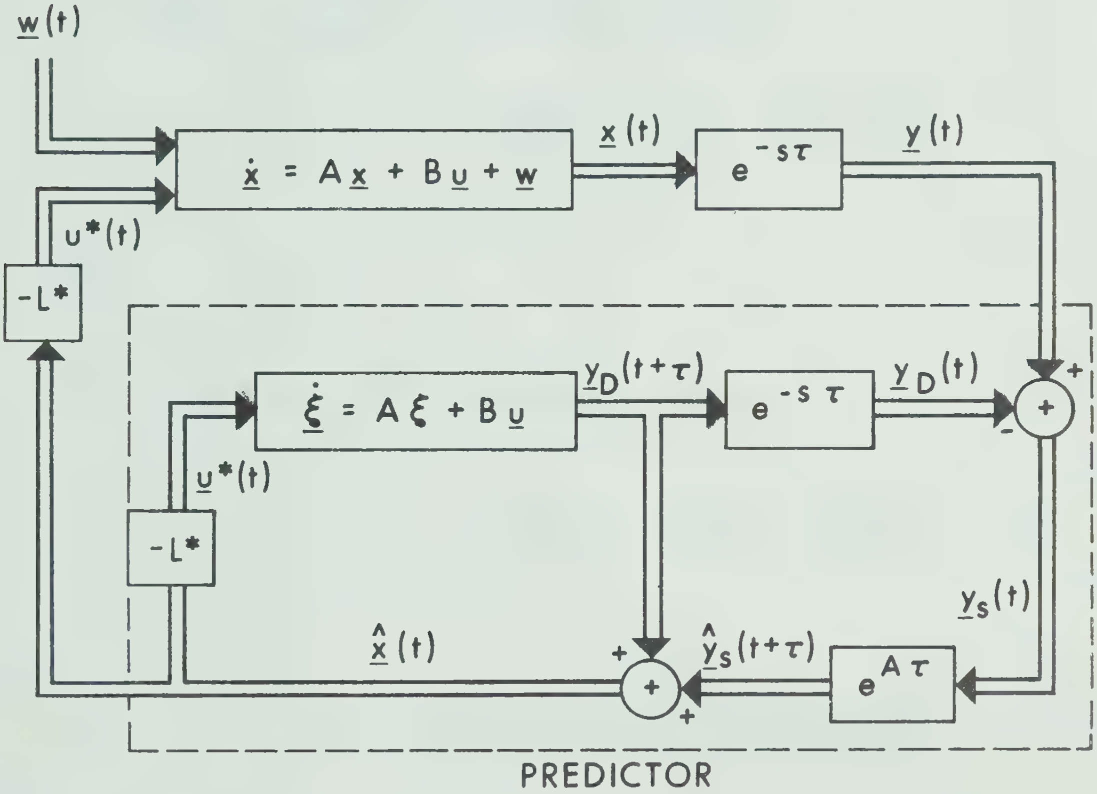

FIGURE 3.1: Optimal stochastic feedback system incorporating a least mean-squared predictor.

3.2 Simulation studies

In this section the performance of the optimal system derived in the previous section will be examined for several values of time delay. Two performance criteria (cost functionals) will be used.[23]

1. $$J_{E1}(\underline{u}) = E\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T [10(\delta w_1^2(t) + \delta w_2^2(t)) + \delta f_1^2(t) + \delta f_2^2(t) + \delta f_3^2(t)]dt\right\}$$

and,

2. $$J_{E2}(\underline{u}) = E\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T [10(\delta w_1^2(t) + \delta w_2^2(t)) + 4(\delta y_1^2(t) + \delta y_2^2(t) + \delta y_3^2(t)) + \delta f_1^2(t) + \delta f_2^2(t) + \delta f_3^2(t)]dt\right\}$$

As was done in chapter two, the basic approach here will be to simulate the optimal system with time delay for the same initial conditions as those used for the deterministic regulator of appendix one. The performance of the optimal stochastic system and of the optimal deterministic system corrupted by driving noise and time

---

[23] Subscripts 1 and 2 are now introduced to differentiate the two cost functionals.

delay are compared with each other and with the ideal deterministic regulator. This is done by examining the graphical display of the response of each system and the computed mean-square deviation of the actual response from the ideal response.

3.2.1 The simulated system

As seen from figure 3.1, design of the least mean-squared predictor essentially consists of a calculation of the matrix exponential, $\exp(A\tau)$, where τ is the feedback time delay. Since the two systems to be simulated differ only in their index of performance the value of $\exp(A\tau)$ will be the same for both and will thus be calculated now.

CALCULATION OF EXP($A\tau$)

The series solution for the exponential of the matrix $A\tau$, given by

$$\ell^{A\tau} = I + (A\tau) + \frac{A\tau}{2}\cdot\left(\frac{A\tau}{1!}\right) + \frac{A\tau}{3}\cdot\left(\frac{(A\tau)^2}{2!}\right) + \ldots\ldots\ldots + \frac{A\tau}{n}\cdot\left(\frac{A^{n-1}\tau^{n-1}}{(n-1)\,!}\right) + \ldots\ldots\ldots,$$

is amenable to computation using a digital computer. A conveneint recursive scheme is provided by noting that each term in parentheses is equal to the entire preceding term; the computation being carried out to only enough terms so that additional terms are negligible

in comparison with the partial sum to that point [35]. Table 3-1 gives computation results for several values of time delay τ.

SYSTEM WITH COST FUNCTIONAL $J_{E1}(\underline{u})$

Cost functional J_{E1} is identical to that used in chapter two and hence the optimal feedback gain matrix L*, given by

$$L^* = R^{-1} B' \hat{K}$$

is identical to that used previously ($\hat{K}$ is given by equation (A1-10) of appendix one).

SYSTEM WITH COST FUNCTIONAL $J_{E2}(\underline{u})$

To place the cost functional in the required form of equation (2-5) requires that the matrices Q and R be specified as

$$Q = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

respectively. Solving for the real symmetric positive definite matrix

TABLE 3-1

Exp(Aτ) for several values of time delay τ

τ (SECS.)	Exp(Aτ)				
0.1	0.9048	0.0	0.0	0.0	0.0
	0.0952	1.0	-0.0952	0.0	0.0
	0.0	0.0	0.9048	0.0	0.0
	0.0	0.0	0.0952	1.0	-0.0952
	0.0	0.0	0.0	0.0	0.9048
0.2	0.8187	0.0	0.0	0.0	0.0
	0.1813	1.0	-0.1813	0.0	0.0
	0.0	0.0	0.8187	0.0	0.0
	0.0	0.0	0.1813	1.0	-0.1813
	0.0	0.0	0.0	0.0	0.8187
0.3	0.7408	0.0	0.0	0.0	0.0
	0.2592	1.0	-0.2592	0.0	0.0
	0.0	0.0	0.7408	0.0	0.0
	0.0	0.0	0.2592	1.0	-0.2592
	0.0	0.0	0.0	0.0	0.7408
0.4	0.6703	0.0	0.0	0.0	0.0
	0.3297	1.0	-0.3297	0.0	0.0
	0.0	0.0	0.6703	0.0	0.0
	0.0	0.0	0.3297	1.0	-0.3297
	0.0	0.0	0.0	0.0	0.6703
0.41	0.6636	0.0	0.0	0.0	0.0
	0.3363	1.0	-0.3363	0.0	0.0
	0.0	0.0	0.6636	0.0	0.0
	0.0	0.0	0.3363	1.0	-0.3363
	0.0	0.0	0.0	0.0	0.6636
0.42	0.6570	0.0	0.0	0.0	0.0
	0.3429	1.0	-0.3429	0.0	0.0
	0.0	0.0	0.6570	0.0	0.0
	0.0	0.0	0.3429	1.0	-0.3429
	0.0	0.0	0.0	0.0	0.6570
0.5	0.6065	0.0	0.0	0.0	0.0
	0.3935	1.0	-0.3935	0.0	0.0
	0.0	0.0	0.6065	0.0	0.0
	0.0	0.0	0.3935	1.0	-0.3935

$$\hat{K} = \lim_{\tau \to \infty} K(\tau)$$

from equation (A1-7), with Q and R as above, gives

$$\hat{K} = \begin{bmatrix} 2.094 & 2.494 & -0.586 & 0.669 & -0.272 \\ 2.494 & 8.967 & -1.826 & 1.675 & -0.668 \\ -0.586 & -1.826 & 2.408 & 1.826 & -0.586 \\ 0.668 & 1.675 & 1.826 & 8.967 & -2.494 \\ -0.272 & -0.668 & -0.586 & -2.494 & 2.094 \end{bmatrix}$$

The optimal feedback gains are then

$$L^* = R^{-1} B' \hat{K}$$

3.2.2 Simulation results

SYSTEM WITH COST FUNCTIONAL $J_{E1}(\underline{u})$

Table 3-2 and table 3-3 give the mean-square deviation of the system variables (versus time delay) for the two cases where : 1. the least mean-squared predictor is excluded from the feedback path and,2. the least mean-squared predictor is included, respectively. A graphical representation of sample results is given in figure 3.2.

Figures 3.3 to 3.6, and figures 3.7 to 3.8 graphically record the response of the system of figure 3.1 for delay times of 0.1 second

and 0.42 second, respectively, when the predictor is not present (i.e. this is equivalently the behaviour of the optimal deterministic regulator corrupted by driving noise and feedback time delay). Similarly, results for the case where the predictor is included are given in figures 3.9 to 3.12. For delays of less than about 0.1 second the system response is seen to be fairly acceptable without a predictor. However, for larger delay times the system becomes quite oscillatory; approaching a limit cycle (for the positional error variables) at about 0.42 second. Figure 3.9 to figure 3.12 clearly depict the usefulness of adding a predictor to those systems with relatively large time delay. It is there shown that a satisfactory response is obtainable at a time delay (0.45 second in this case) which would otherwise result in an unstable system.

SYSTEM WITH COST FUNCTIONAL $J_{E2}(\underline{u})$

Table 3-4 and 3-5 give the mean-square deviation of the system variables (versus time delay) for the two cases where: 1. the least mean-squared predictor is excluded from the feedback path, and 2. the least mean-squared predictor is included, respectively. A graphical representation of sample results is given in figure 3.13.

Here too, the response of the system for time delays of less than about 0.1 second is fairly acceptable without a predictor. However, figure 3.14 and figure 3.15 now show that (unlike the system with cost J_{E1}) the response at a time delay of 0.42 second is unstable.

TABLE 3-2

Mean-square deviations without predictor : cost J_{E1} *

τ (SECS.)	MEAN-SQUARE DEVIATIONS		
	x	y	u
0.1	0.0294	0.0855	
	0.0127	0.3978	1.9438
	0.0421	0.1227	3.2342
	0.0040	0.1040	0.1785
	0.0035	0.0078	
0.2	0.1491	0.3484	
	0.0679	0.8942	4.6635
	0.2515	0.5428	8.2186
	0.0302	0.2491	0.5921
	0.0257	0.0417	
0.3	0.5067	0.9017	9.4438
	0.2876	1.6040	18.8290
	1.0426	1.6234	2.0157
	0.1713	0.5284	
	0.1470	0.1819	
0.4	1.9480	2.5325	
	1.5930	3.2909	24.4114
	5.3503	6.1294	64.4490
	1.3025	1.6685	11.3170
	1.0489	1.0776	
0.42	2.6242	3.2350	
	2.3255	3.9456	30.8118
	7.5537	8.3344	85.9792
	1.9897	2.1903	16.0981
	1.5450	1.5615	
0.50	9.2087	7.3013	
	8.1656	9.8276	75.3953
	30.3350	20.3079	232.9885
	7.4212	7.2480	47.4542
	6.8476	3.9877	

* The MSD for the time delay problem is calculated by comparing the deterministic and stochastic model results at the corresponding iteration cycles.

TABLE 3-3

Mean-square deviations with predictor : cost J_{E1}

τ (SECS.)	MEAN-SQUARE DEVIATIONS				
	$\underline{x}$	$\underline{y}$	$\underline{\hat{x}}$	$\underline{u}$	$\underline{\hat{x}}$ †
0.1	0.0006 0.0016 0.0005 0.0007 0.0008	0.0263 0.3770 0.0361 0.0979 0.0029	0.0004 0.0013 0.0003 0.0006 0.0006	0.0057 0.0064 0.0029	0.0002 0.0002 0.0002 0.0002 0.0003
0.2	0.0007 0.0018 0.0005 0.0008	0.0888 0.7760 0.1241 0.2025	0.0004 0.0001 0.0003 0.0007	0.0055 0.0065 0.0029	0.0004 0.0005 0.0003 0.0005
0.3	0.0008 0.0020 0.0005 0.0010 0.0008	0.1756 1.1753 0.2451 0.3066 0.0116	0.0003 0.0012 0.0003 0.0008 0.0004	0.0053 0.0065 0.0031	0.0006 0.0008 0.0004 0.0006 0.0006
0.4	0.0010 0.0022 0.0005 0.0011 0.0007	0.2762 1.5581 0.3829 0.4046 0.0167	0.0003 0.0010 0.0003 0.0008 0.0004	0.0048 0.0058 0.0032	0.0007 0.0011 0.0004 0.0007 0.0007
0.50	0.0012 0.0024 0.0006 0.0013 0.0007	0.3831 1.9132 0.5258 0.4930 0.0219	0.0003 0.0010 0.0003 0.0009 0.0003	0.0046 0.0056 0.0033	0.0008 0.0014 0.0004 0.0007 0.0007

† Comments in the footnote of table 2-4 apply.

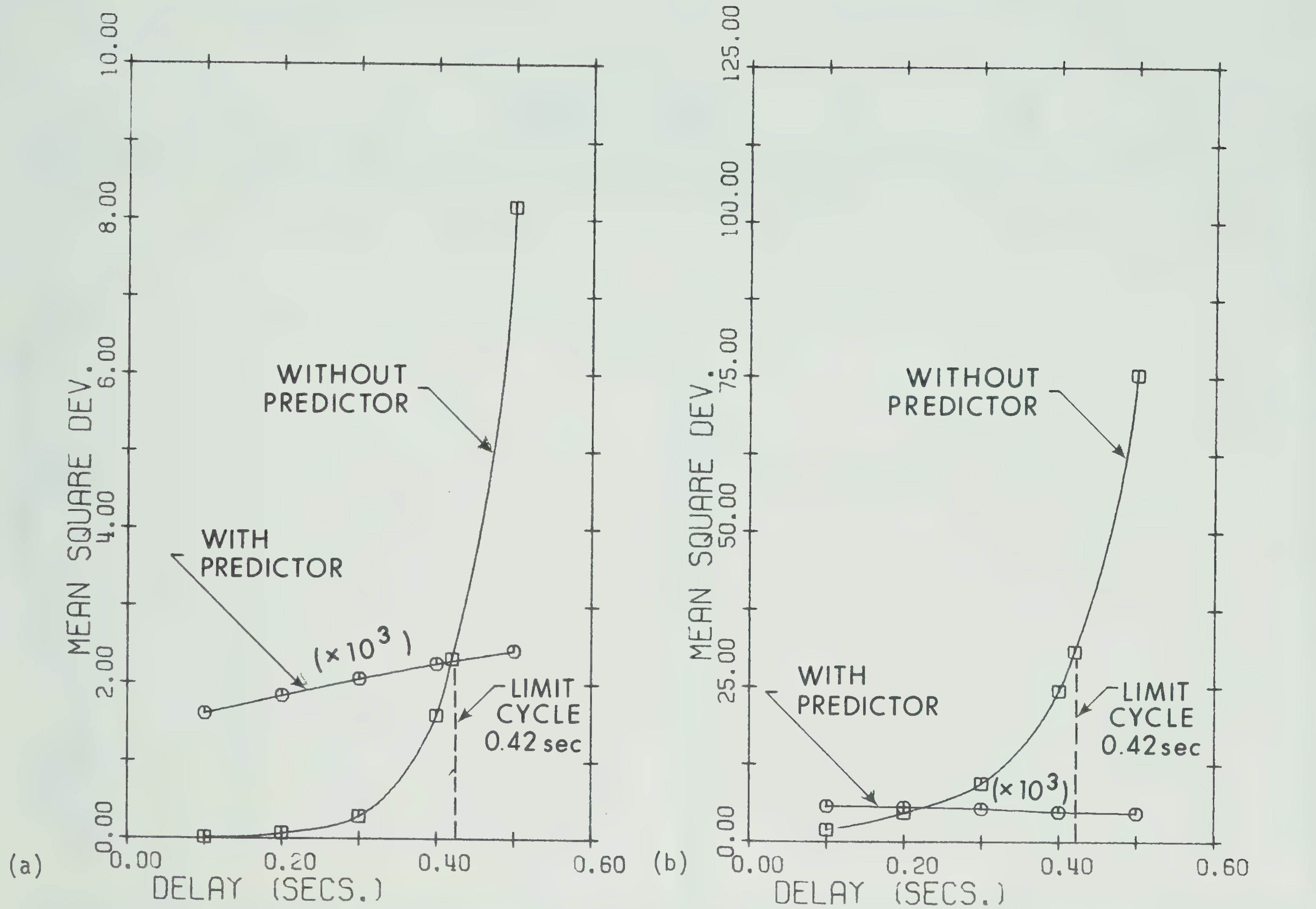

FIGURE 3.2: Mean-square deviations for (a) the positional error variable δw_1, and (b) the corrective force δF_1. Cost J_{E2}.

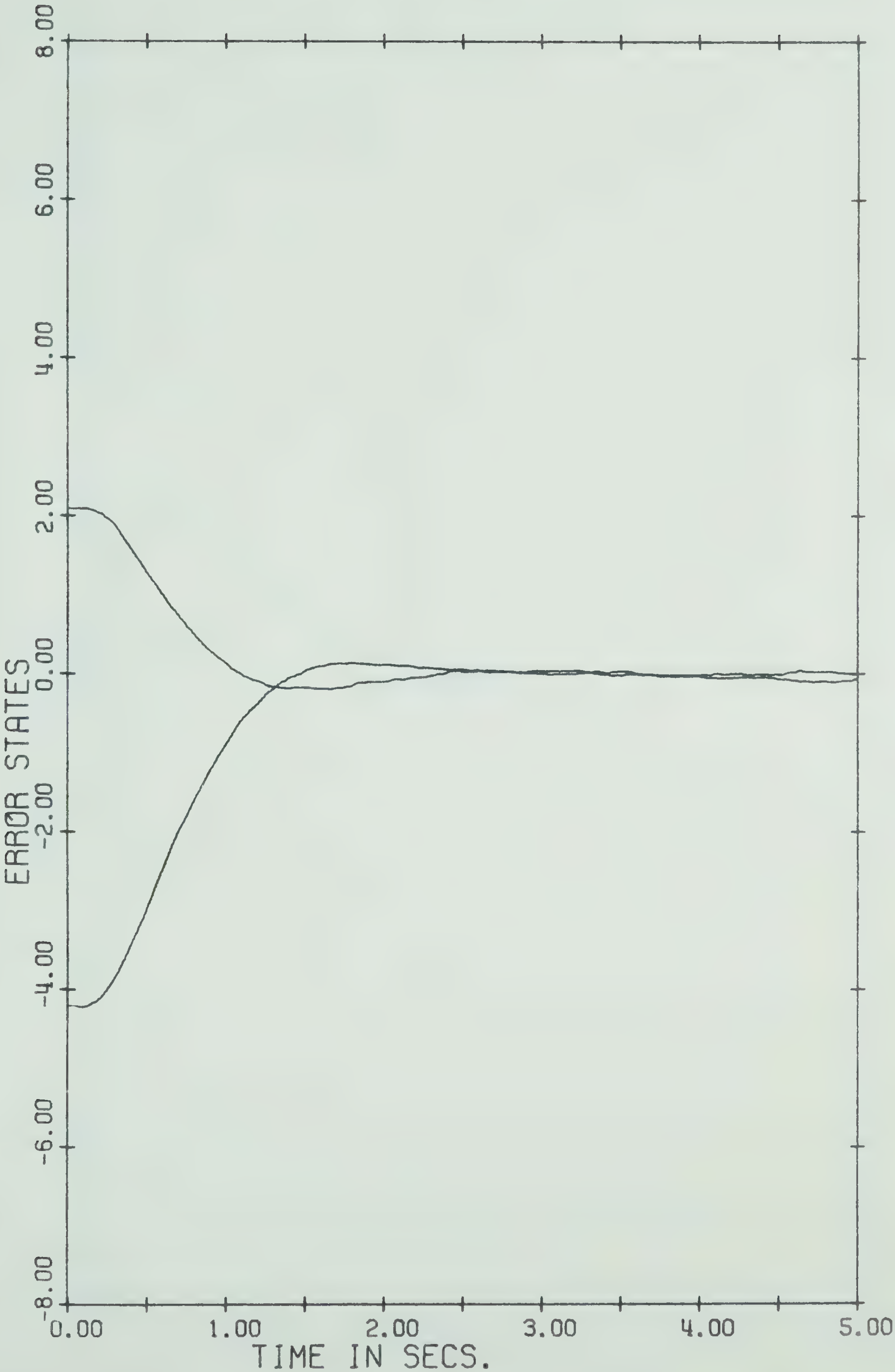

FIGURE 3.3: Position deviations with no predictor: τ=0.1 second.
Cost J_{E1}.

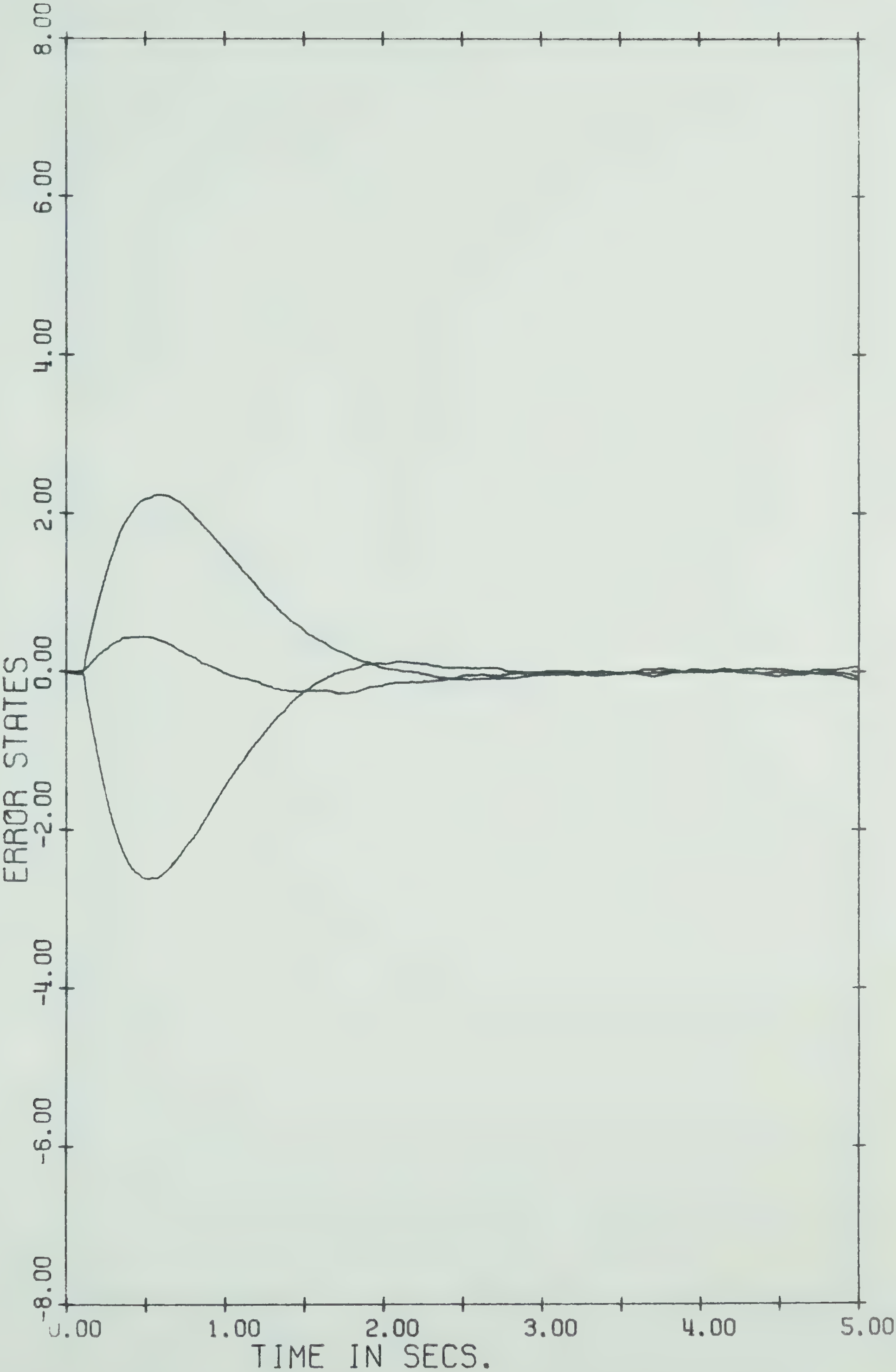

FIGURE 3.4: Velocity deviations with no predictor τ=0.1 second. Cost J_{E1}.

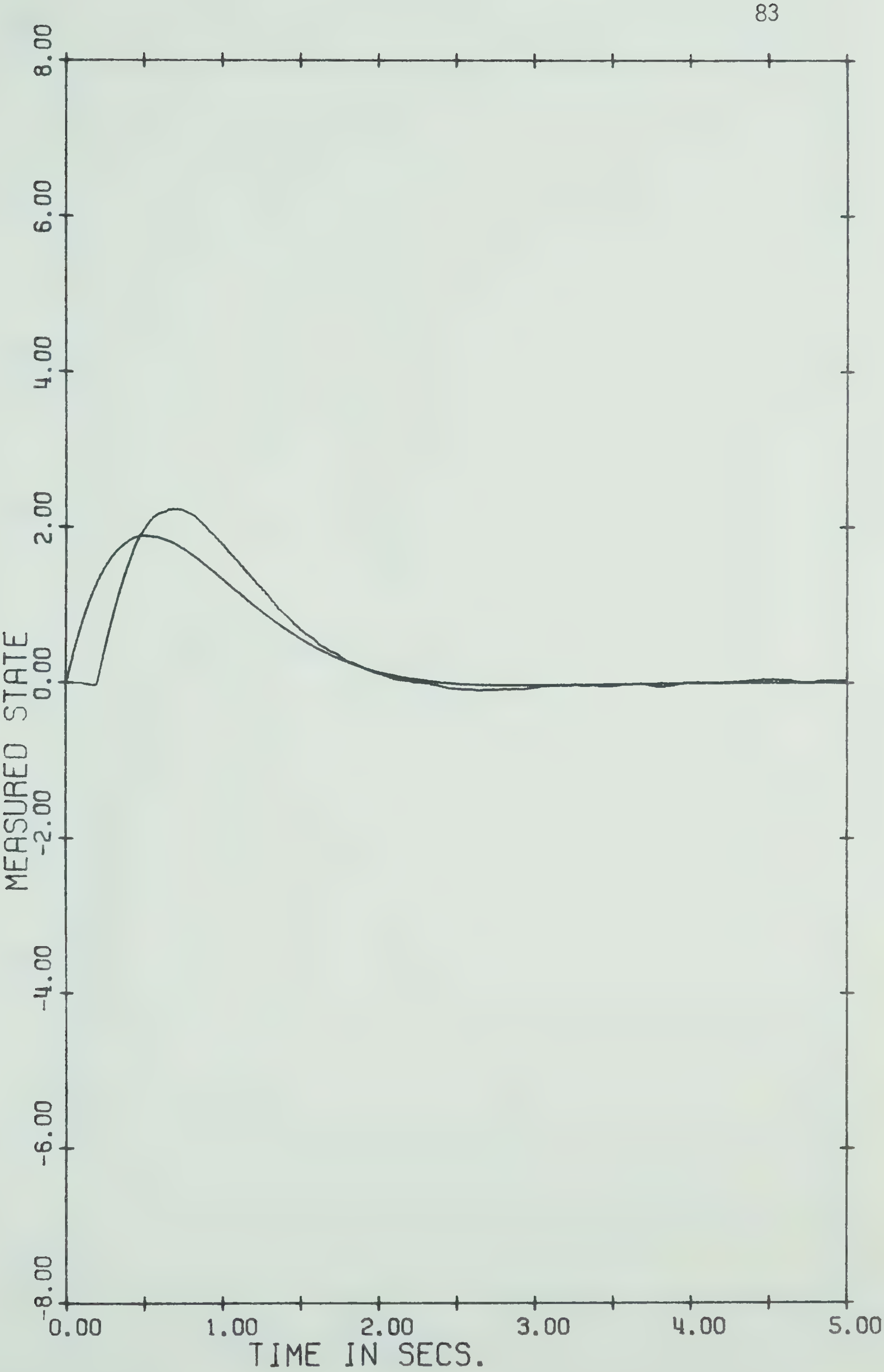

FIGURE 3.5: Measured state of δy_1 with no predictor: τ=0.1 second. Cost J_{E1}.

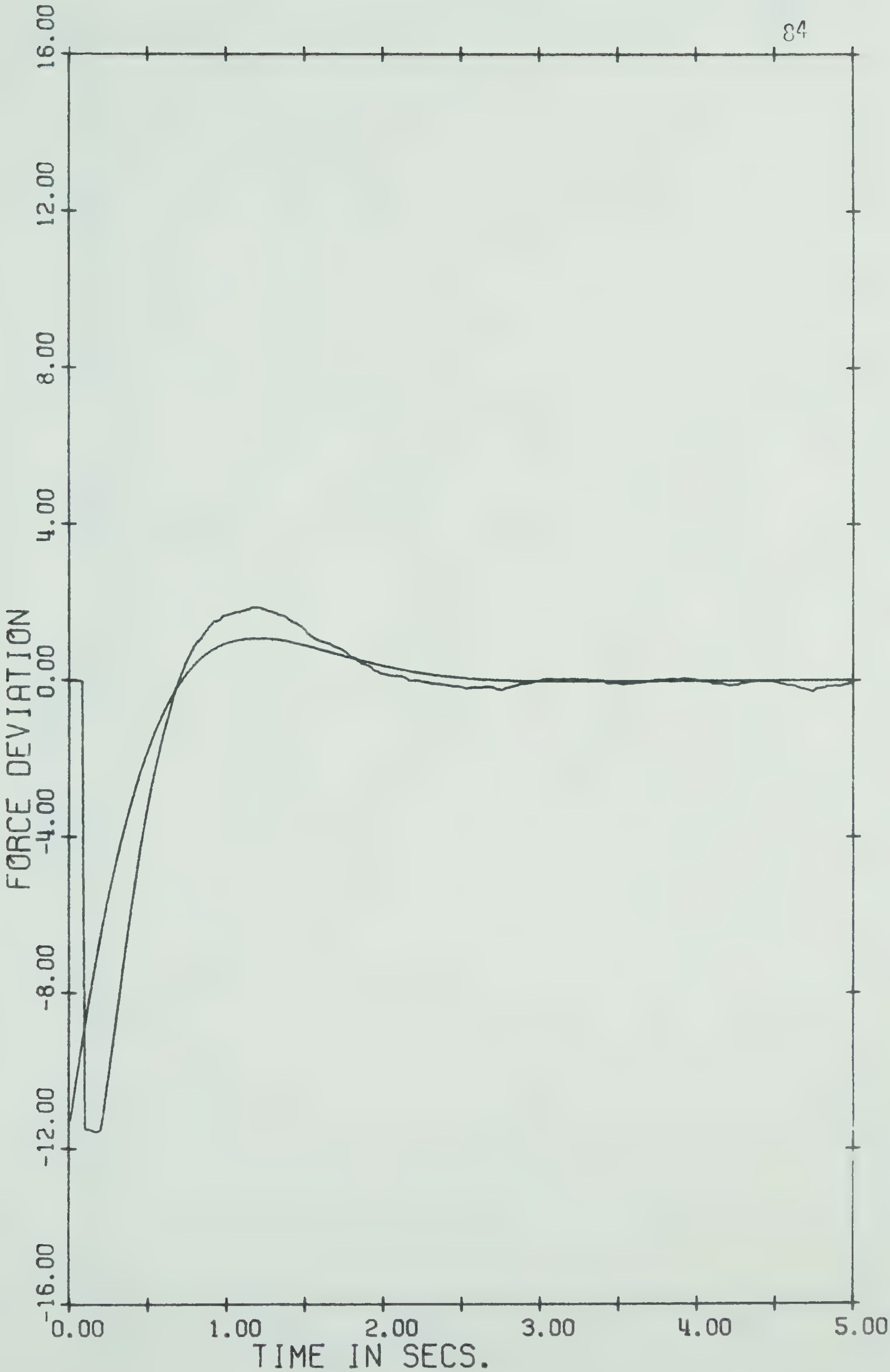

FIGURE 3.6: Control for second vehicle (δF_2) with no predictor: τ=0.1 second. Cost J_{E1}.

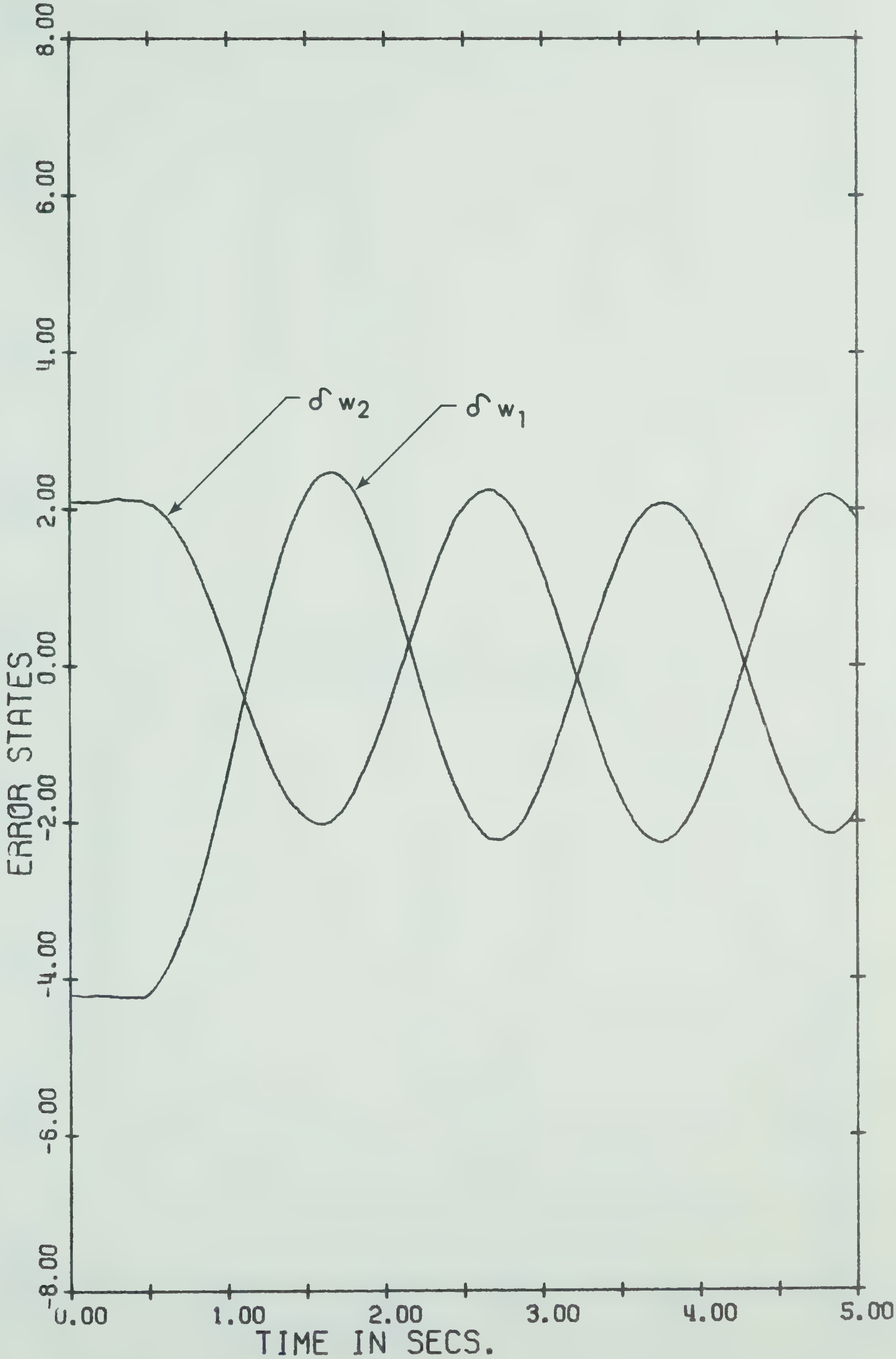

FIGURE 3.7: Position deviations with no predictor: τ=0.42 second. Cost J_{E1}.

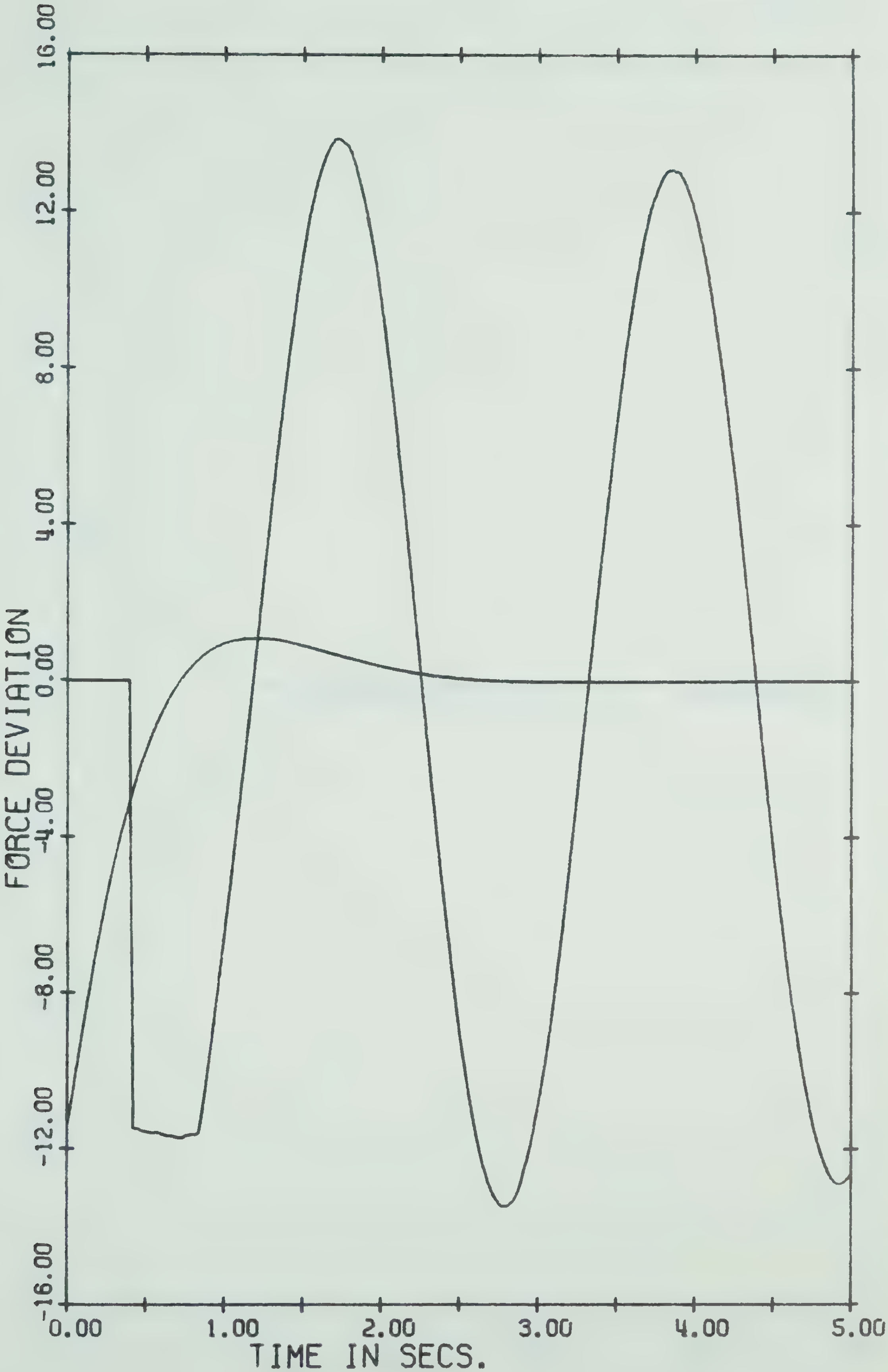

FIGURE 3.8: Control for second vehicle (δF_2) with no predictor: τ=0.42 second. Cost J_{E1}.

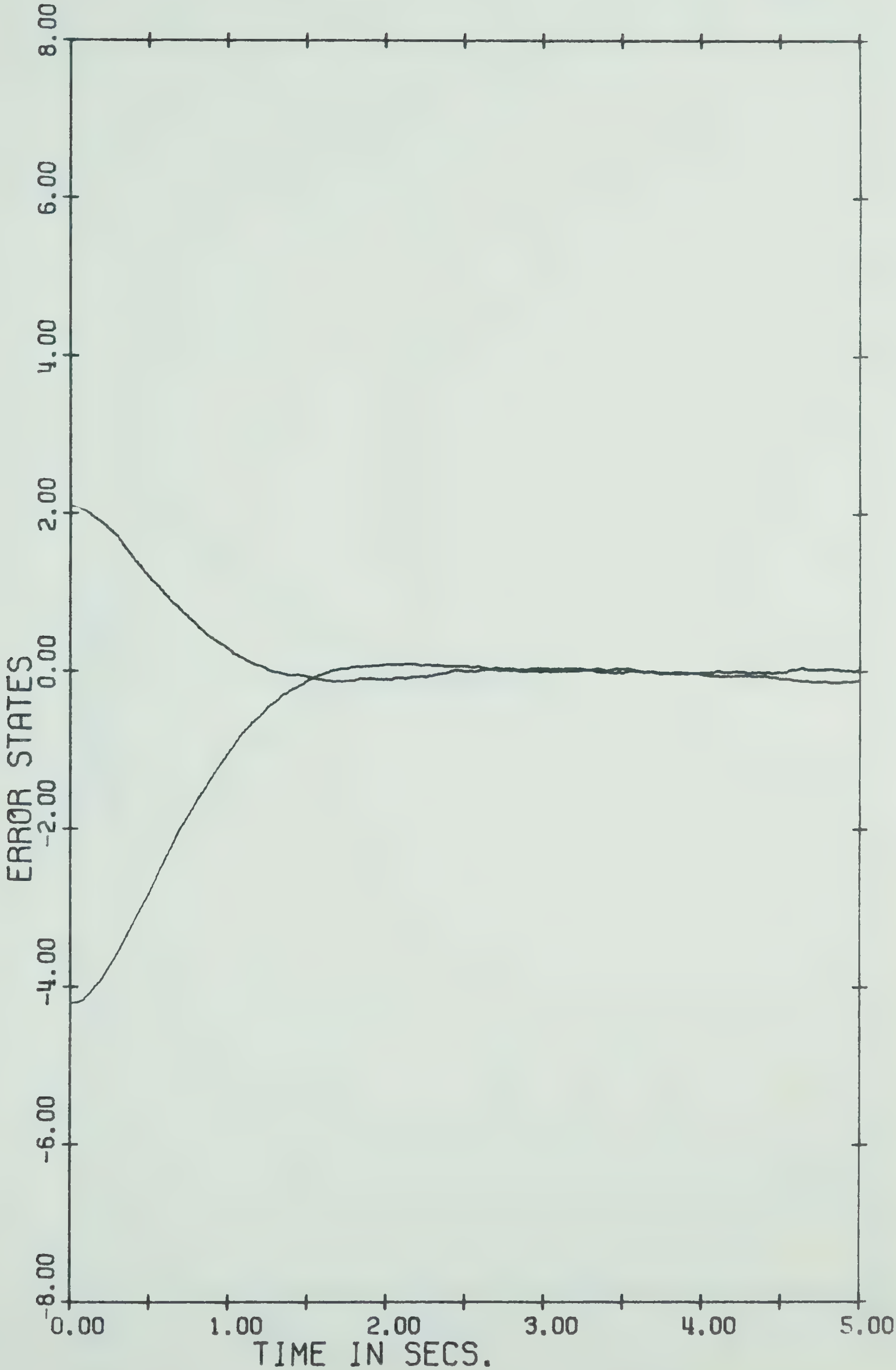

FIGURE 3.9: Position deviations with predictor: τ=0.45 second. Cost J_{E1}.

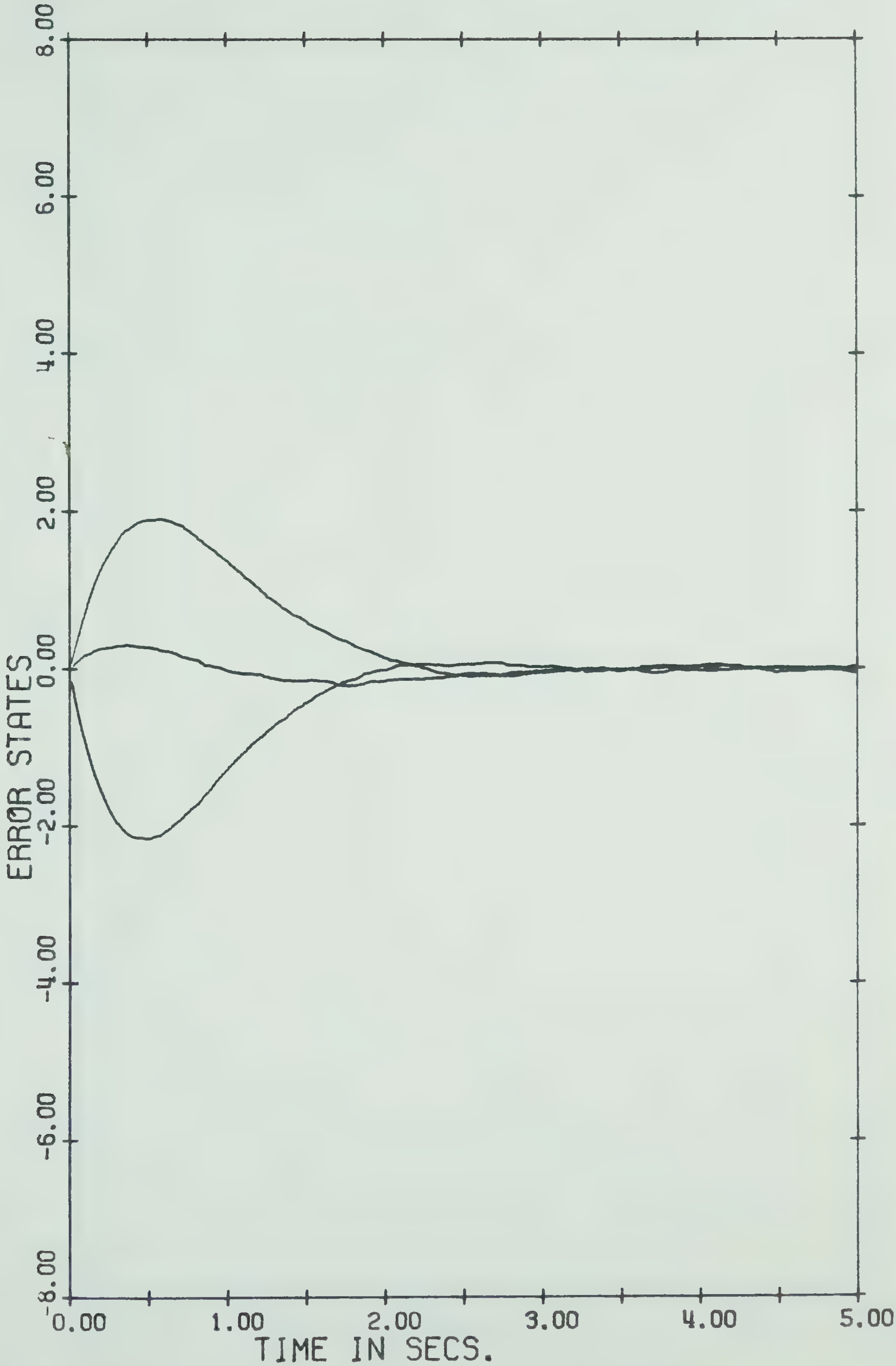

FIGURE 3.10: Velocity deviations with predictor: τ=0.45 second.
Cost J_{E1}.

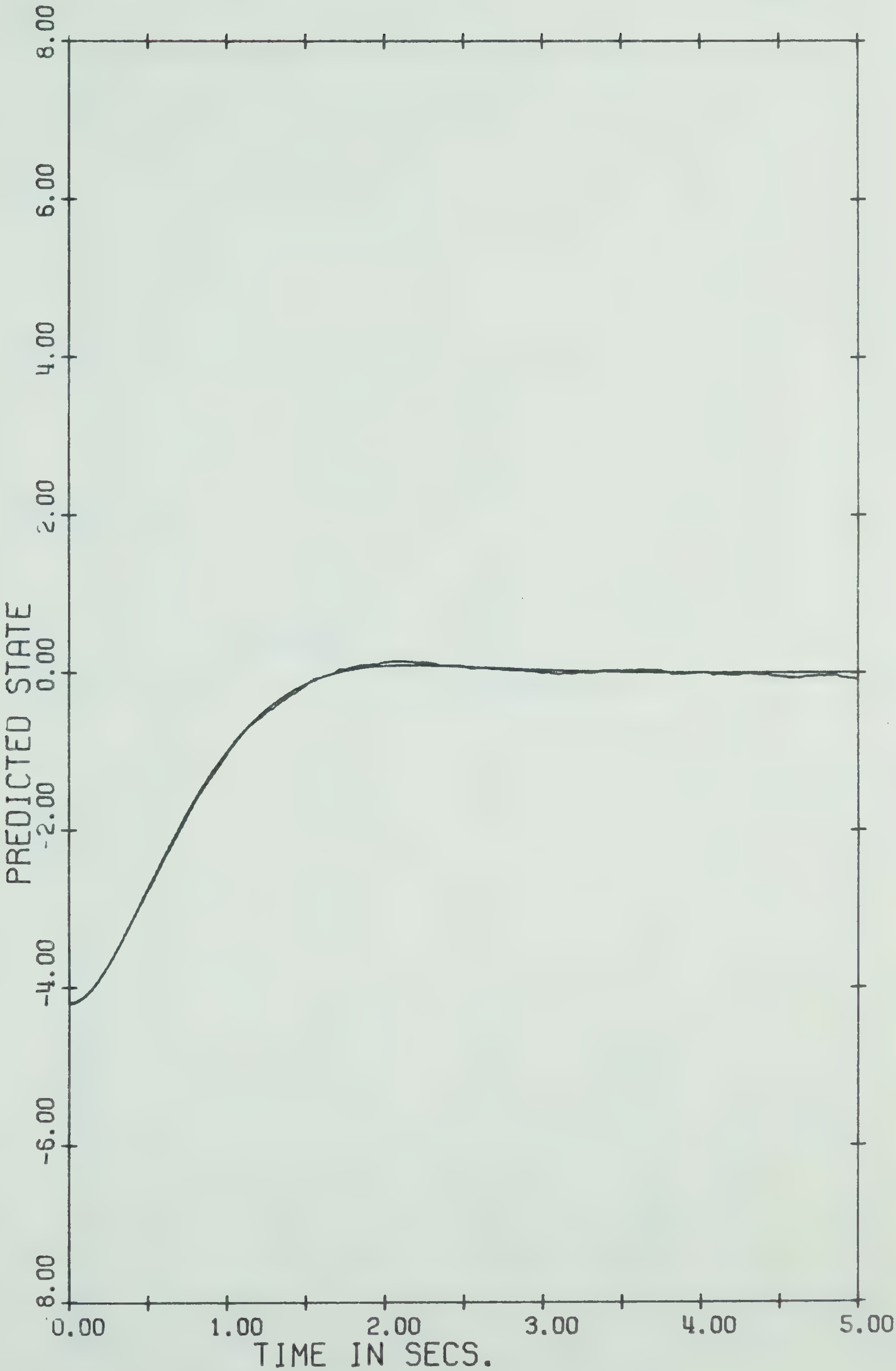

FIGURE 3.11: Predicted state δW_1 with predictor: τ=0.45 second. Cost J_{E1}.

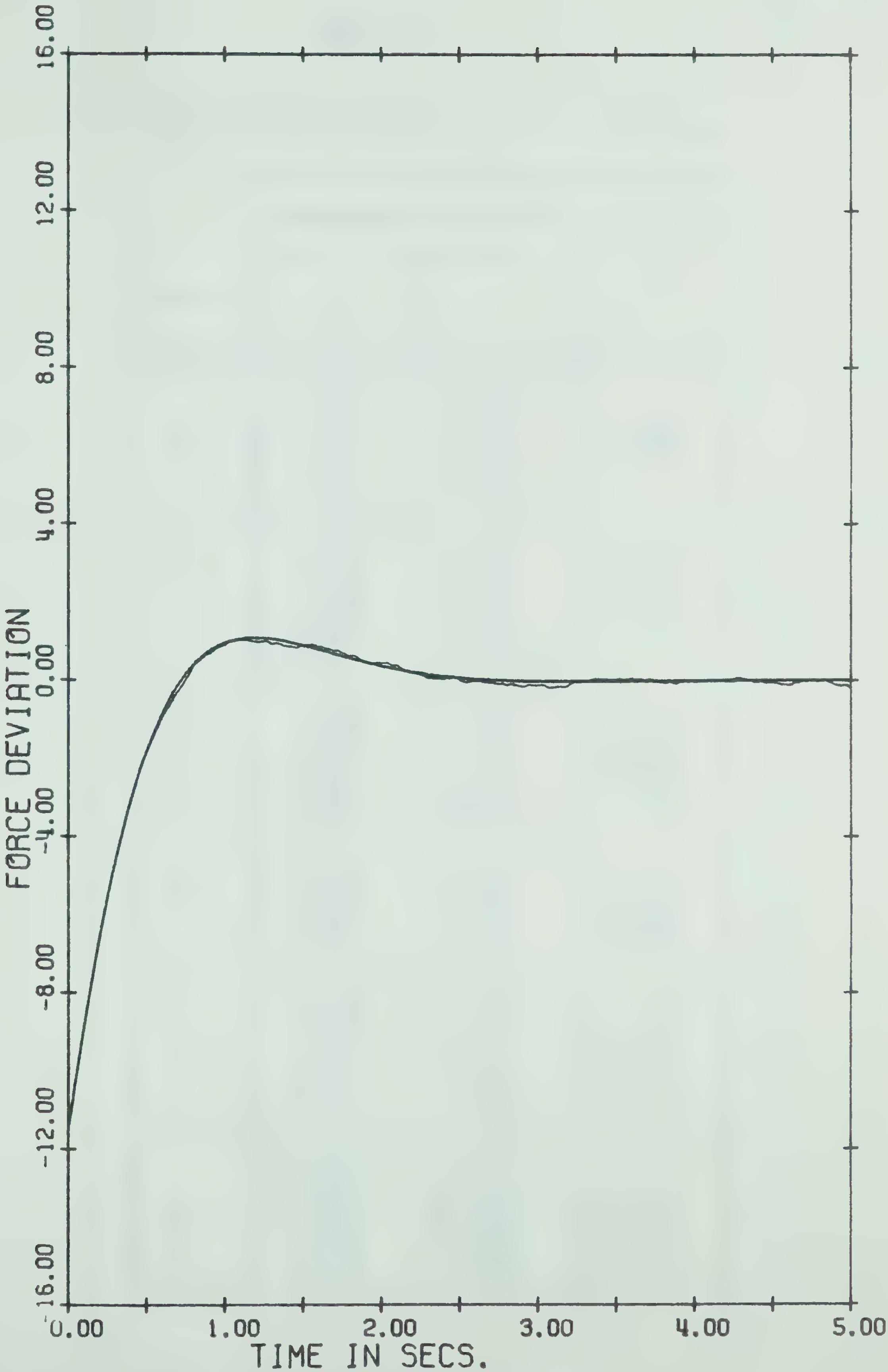

FIGURE 3.12: Control for second vehicle (δF_2) with predictor: τ=0.45 second. Cost J_{E1}.

TABLE 3-4

Mean-square deviations without predictor : cost J_{E2}

τ (SECS.)	MEAN-SQUARE DEVIATIONS		
	x	y	u
0.1	0.0032 0.0086 0.0335 0.0025 0.0026	0.0678 0.3869 0.1000 0.1008 0.0060	1.9396 3.1897 0.1715
0.2	0.1222 0.0446 0.2020 0.0184 0.0173	0.2762 0.8404 0.4335 0.2289 0.0290	4.6345 7.9990 0.5263
0.3	0.4359 0.1988 0.8637 0.1085 0.1016	0.7326 1.4405 1.3112 0.4428 0.1248	9.5530 18.4956 1.7526
0.4	2.1055 1.3087 5.6404 0.9671	2.3719 2.9932 5.6229 1.4137	28.4629 72.0096 11.1904
0.41	2.5208 1.6258 6.9226 1.2267 1.2113	2.7210 3.3286 6.5994 1.6668 1.0186	33.0200 86.0017 13.8811
0.5	10.6833 10.2259 31.6997 8.7400 5.9952	9.4514 8.7065 26.7356 5.8070 4.8698	111.0299 333.9804 63.5443 63.5443

TABLE 3-5

Mean-square deviations with predictor: cost J_{E2}

τ (SECS.)	MEAN-SQUARE DEVIATIONS				
	$\underline{x}$	$\underline{y}$	$\underline{\hat{x}}$	$\underline{u}$	$\underline{\hat{x}}$ †
0.1	0.0005	0.0217	0.0003	0.0063	0.0002
	0.0016	0.3732	0.0014	0.0058	0.0002
	0.0004	0.3100	0.0003	0.0035	0.0002
	0.0007	0.0970	0.0006		0.0002
	0.0006	0.0023	0.0004		0.0003
0.2	0.0007	0.0888	0.0004		0.0004
	0.0018	0.7760	0.0013	0.0055	0.0005
	0.0005	0.1241	0.0003	0.0065	0.0003
	0.0008	0.2025	0.0007	0.0029	0.0005
	0.0008	0.0067	0.0005		0.0005
0.3	0.0007	0.1412	0.0002		0.0006
	0.0020	1.1458	0.0012	0.0056	0.0008
	0.0005	0.2032	0.0002	0.0059	0.0004
	0.0010	0.2997	0.0008	0.0036	0.0007
	0.0006	0.0096	0.0003		0.0006
0.4	0.0010	0.2762	0.0003		0.0007
	0.0022	1.5581	0.0010	0.0048	0.0010
	0.0005	0.3829	0.0003	0.0058	0.0004
	0.0011	0.4046	0.0008	0.0032	0.0007
	0.0008	0.0167	0.0004		0.0007
0.5	0.0011	0.3015	0.0002		0.0008
	0.0024	1.8493	0.0009	0.0045	0.0014
	0.0006	0.4273	0.0002	0.0051	0.0004
	0.0013	0.4795	0.0009	0.0036	0.0007

† Comments in the footnote of table 2-4 apply.

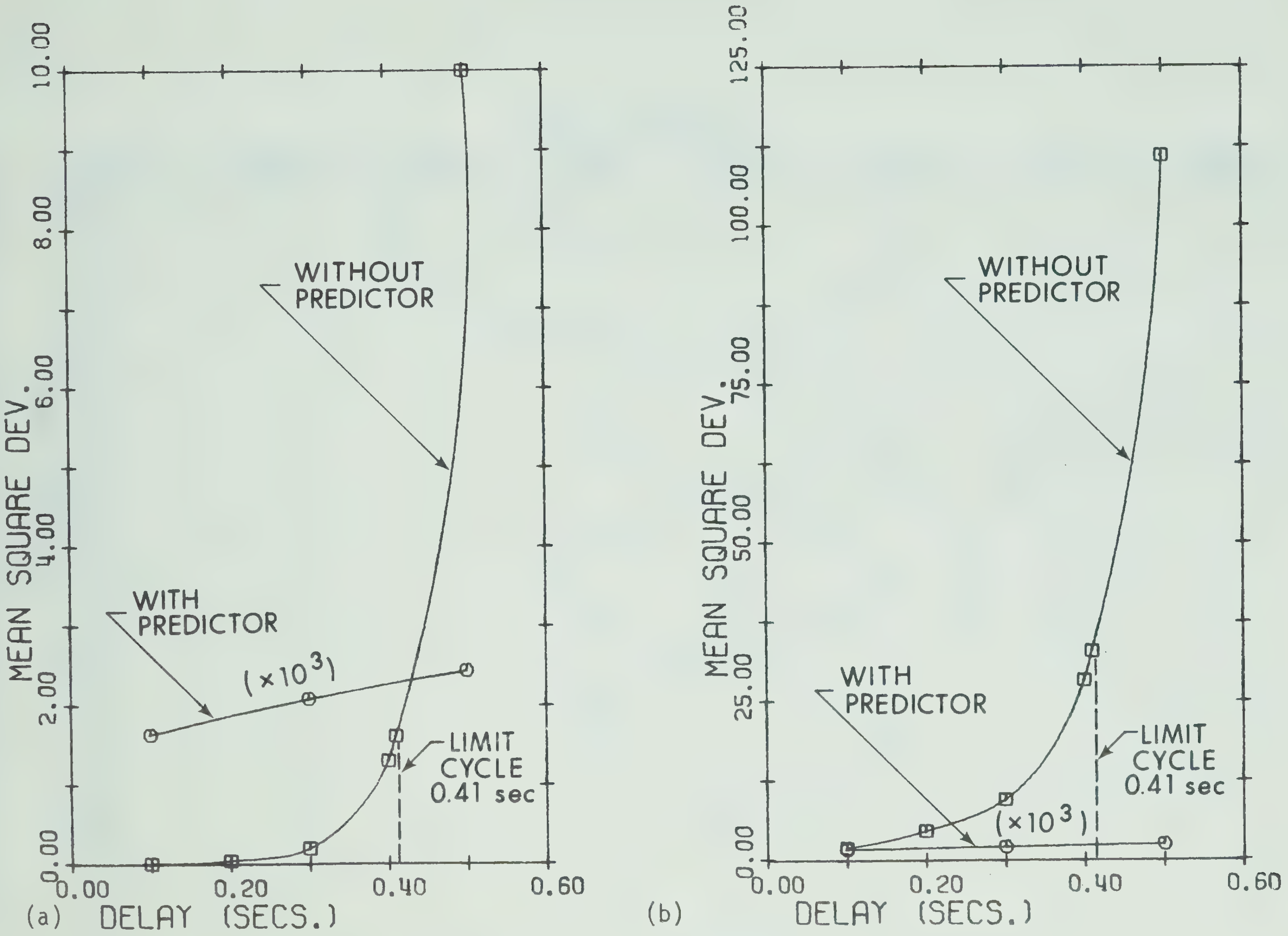

FIGURE 3.13: Mean-square deviations for (a) the positional error variable δw_1 and (b) the corrective force δF_1. Cost ${}^1J_{E2}$.

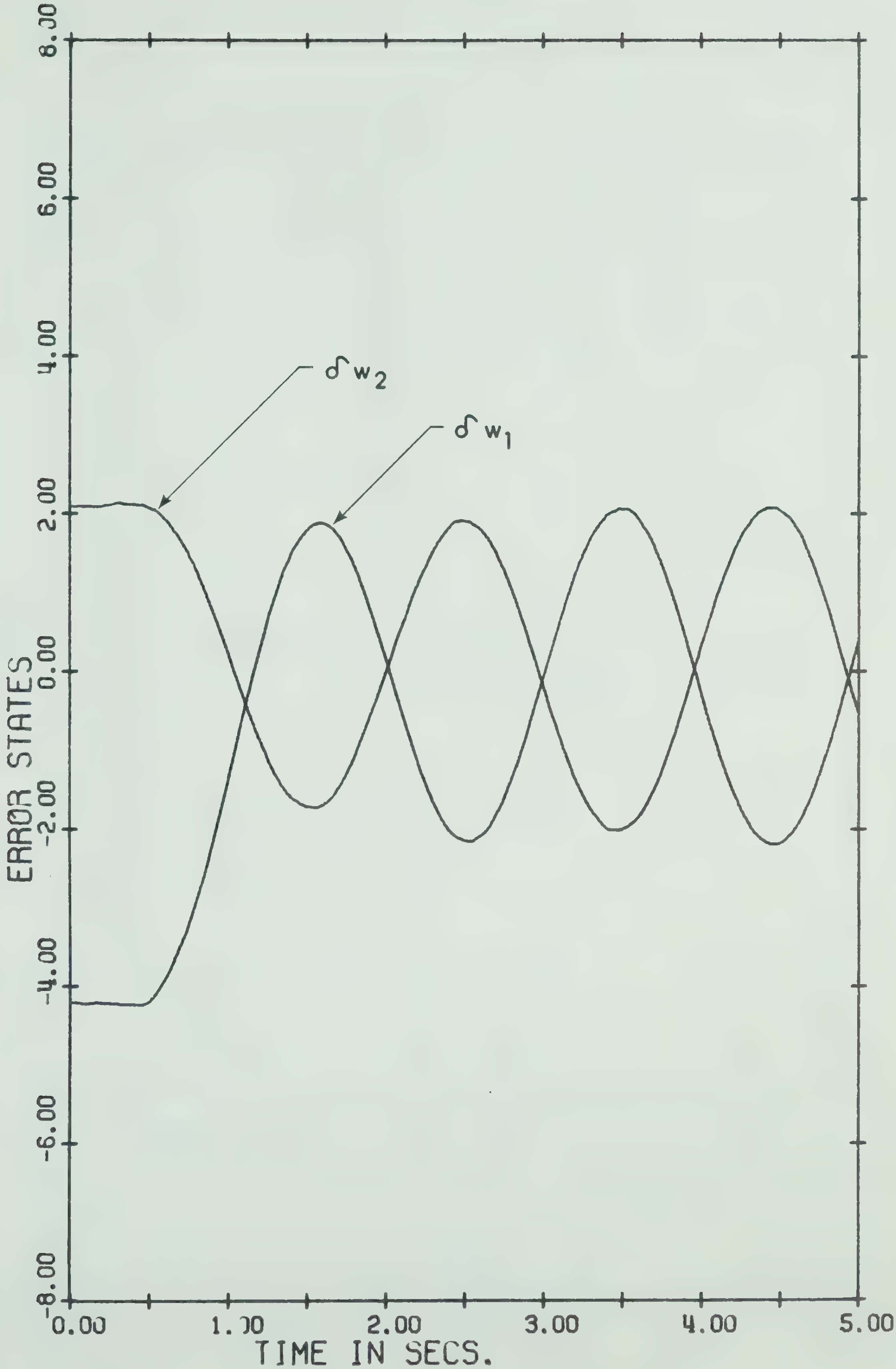

FIGURE 3.14: Position deviations with no predictor: τ=0.42 second. Cost J_{E2}.

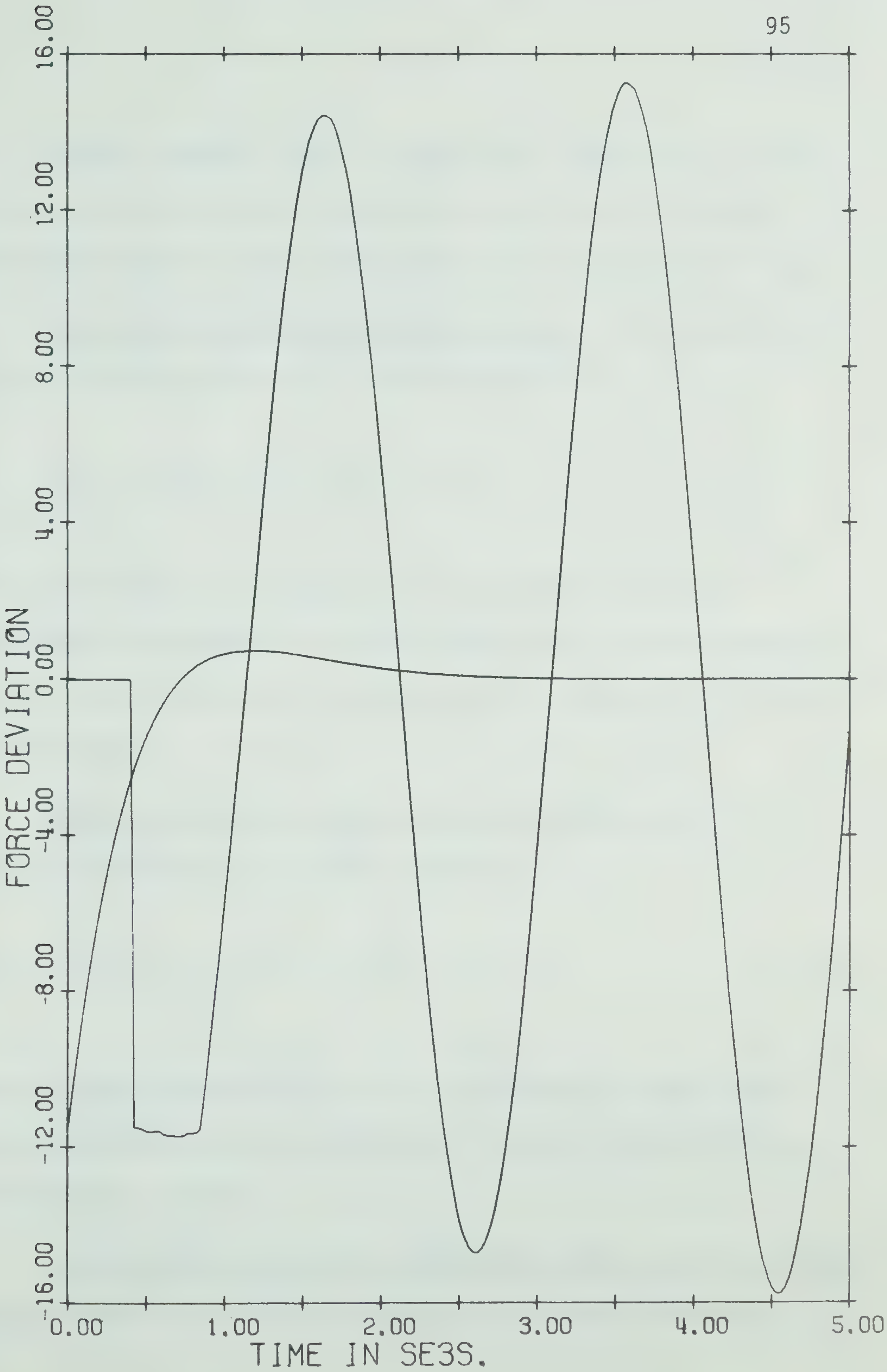

FIGURE 3.15: Control for second vehicle (δF_2) with no predictor: τ=0.42 second. Cost J_{E2}.

Several phenomena that are noted from a comparison of table 3-5 with table 3-3, and of figure 3.14 with figure 3.7 are now mentioned; together with a simple theoretical explanation of each.

<u>The difference in deterministic response</u> - The deterministic response of the regulator is, from equation (3-6),

$$\dot{\underline{y}}_D(t) = A\,\underline{y}_D(t) + B\,\underline{u}\,(t-\tau)$$

Since the feedback gain matrix, L*, that relates the control variables to the measured variables depends upon the specified cost function, the deterministic response, $\underline{y}_D(t)$, changes with the index of performance.

<u>The mean-square deviations of the predicted state</u> - It can easily be shown that, for either regulator,

$$(\hat{\underline{x}}(t) - \underline{x}_D(t)) = + \int^{t} \ell^{A\tau}\,[A\,\underline{y}_S(\sigma) + \underline{w}(\sigma-\tau)]d\sigma \qquad (3\text{-}16)$$

The mean-square deviation of each of the predicted states should thus be comparable for both regulators. Table 3-5 and table 3-3 bear evidence to this.

<u>The mean-square deviations between the actual and the predicted state</u> - Here, it can also be shown that, for either regulator,

$$(\hat{\underline{x}}(t) - \underline{x}(t)) = -\,\underline{x}_S(t) + \int^{t} \ell^{A\tau}[A\underline{y}_S(\sigma) + \underline{w}(\sigma-\tau)]\; d\sigma \qquad (3\text{-}17)$$

The mean-square deviation between each of the actual states and the corresponding predicted states should then also be comparable for both systems. Table 3-5 and table 3-3 show that this is the case.[24]

Comparing equation (3-17) with equation (3-16) it is also evident that, for either regulator, the predicted states should approximate the deterministic regulator states rather more closely than the actual states of the system since

$$[(\hat{\underline{x}}-\underline{x}) - (\hat{\underline{x}}-\underline{x}_D)] = -\,(\underline{x}-\underline{x}_D) = -\,\underline{x}_S(t) = \int^{t} [A\underline{x}_S(\sigma) + \underline{w}(\sigma)]\; d\sigma$$

Comparing column six with column four of table 3-5 (or table 3-3) again shows this to be true.

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The exact agreement in MSD noted here in table 3-5 and table 3-3 (unlike that for the MSD of the predicted states) is most likely due to the way the IBM/System 360 generates the random noise sequence. If the noise sequence generated differs from one simulation run to the next, but if $\underline{w}(t)$ has some fixed relation to $\underline{w}(t-\tau)$, it is possible that, whereas the value of equation (3-16) may vary at each simulation run, that of equation (3-17) can remain fixed.

3.3 Predictor performance when the time delay is not exactly known

Let $\Phi(t,t_o)$ be the state transition matrix of the linear homogeneous vector matrix differential equation

$$\dot{\underline{x}}(t) = A \ \underline{x}(t)$$

where A is the constant matrix of the plant. Then, from equation (3-1)

$$\underline{y}(t) = \underline{x}(t-\tau)\Phi(t-\tau,\ t_o)\ \underline{x}(t_o) + \int_{t_o}^{t-\tau} \Phi(t-\tau,\sigma)\ B\ \underline{u}(\sigma)d\sigma$$

$$+ \int_{t_o}^{t-\tau} \Phi(t-\tau,\sigma)\ \underline{w}(\sigma)d\sigma;\ t-\tau \geq t_o \qquad (3\text{-}18)$$

Let the assumed system delay be τ_p such that

$$\underline{y}_{DP}(t) = \underline{x}_{DP}(t-\tau_p) \qquad (3\text{-}19)$$

where $\underline{x}_{DP}(t)$ and $\underline{y}_{DP}(t)$ are the state and output vectors of the deterministic portion of the filter model, respectively. Then,

$$\underline{y}_{DP}(t) = \underline{x}_{DP}(t-\tau_p) = \Phi(t-\tau_p,t_o)\ \underline{x}_{DP}(t_o)$$

$$+ \int_{t_o}^{t-\tau_p} \Phi(t-\tau_p,\sigma)\ B\ \underline{u}(\sigma)d\sigma;\ t-\tau_p \geq t_o \qquad (3\text{-}20)$$

where $\underline{x}_{DP}(t_o)$ are the initial states of the filter plant model.

Since $\underline{x}_{DP}(t_o)$ and $\underline{x}(t_o)$ are set such that

$$\underline{x}_{DP}(t_o) = \underline{x}(t_o)$$

then it can be shown that, if $\tau_P=\tau$, $\underline{y}_S(t)$ (shown in figure 3.1) is given by

$$\underline{y}_S(t) = \underline{y}(t) - \underline{y}_{DP}(t) = \int_{t_o}^{t-\tau} \Phi(t-\tau,\sigma)\ \underline{w}(\sigma)d\sigma ; t-\tau \geq t_o \tag{3-21}$$

$\underline{y}_S(t)$ is the result of a linear operation on the Gaussian white noise vector $\underline{w}(t)$ and hence, $\underline{y}_S(t)$ is also a purely Gaussian white noise vector and equation (3-13) is valid [13, 23]. If, however, the system delay, τ, is not exactly known but is assumed to be

$$\tau_P = \tau + \tau_\varepsilon$$

where τ_ε is the difference between the assumed and actual delay, then it can be easily shown that

$$\underline{y}_S(t) = \underline{y}(t) - \underline{y}_{DP}(t) = \underline{x}(t-\tau) - \underline{x}_{DP}(t-\tau_p)$$

$$= [\Phi(t-\tau,t_o) - \phi((t-\tau) - \tau_\varepsilon,t_o)]\ \underline{x}(t_o)$$

$$+ \{ \int_{t_o}^{t-\tau} \phi(t-\tau,\sigma)B\underline{u}(\sigma)d\sigma - \int_{t_o}^{(t-\tau)-\tau_\varepsilon} \Phi((t-\tau)-\tau_\varepsilon,\zeta)B\ \underline{u}\ (\zeta)\ d\zeta\ \}$$

$$+ \int_{t_o}^{t-\tau} \Phi(t-\tau,\sigma)\underline{w}\ (\sigma)\ d\sigma\ ;\ t-\tau_p \geqslant t_o,\ \text{if}\ \tau_\varepsilon \geqslant 0$$

$$t-\tau \geqslant t_o,\ \text{if}\ \tau_\varepsilon < 0 \qquad (3\text{-}22)$$

Unlike the previous case where τ_p was equal to τ, $\underline{y}_S(t)$ now contains a deterministic component due to : (i) the difference in propagation of initial conditions (first term on right hand side of (3-22)), and (ii) the difference in applied control action (second term on the right hand side of (3-22)), in addition to the Gaussian white noise of (3-21). Hence, the statement of equation (3-13)is here entirely inappropriate and

$$E[\underline{y}_S(t+\tau)/\underline{y}_S(\sigma),\ \sigma \leq t] \neq \ell^{A\tau}\ \underline{y}_S(t)$$

From an examination of equation (3-22) and figure 3.1 several comments can be made about the possible behaviour of the system when

the filter has an inaccurate estimate of the system time delay.

1. For $t \leq \tau$, τ_p (whichever is smaller) only the deterministic portion of the vehicle response is controlled. Random disturbances are completely unaffected by the control effort.

2. The system with a predictor having incorrect information on the system time delay does not perform in the same fashion as that without a predictor and with a time delay equal to the difference between the assumed and actual delay. This is easily seen by noting that in the latter case

$$\underline{u}^*(t) = - L^* \underline{y}(t) = - L^* \underline{x}\,(t-\tau_\varepsilon)$$

$$= - L^* [\, \underline{x}_D(t-\tau_\varepsilon) + \underline{y}_S(t)] \qquad (3\text{-}23)$$

where $\underline{y}_S(t)$ is here the stochastic component of $\underline{y}(t)$. In the former case

$$\underline{u}^*(t) = - L^* \hat{\underline{x}}(t) = - L^* [\underline{x}_D(t) + \ell^{A\tau}\underline{y}_S(t)] \qquad (3\text{-}24)$$

where $\underline{y}_S(t)$ is now given by equation (3-22).

3. For $t > \tau, \tau_p$ (whichever is larger) filter behaviour vastly deteriorates and, unless $|\tau_\varepsilon|$ is small, system performance may be better without it.

Figures 3.16 to 3.19 prove the correctness of the forementioned notes.

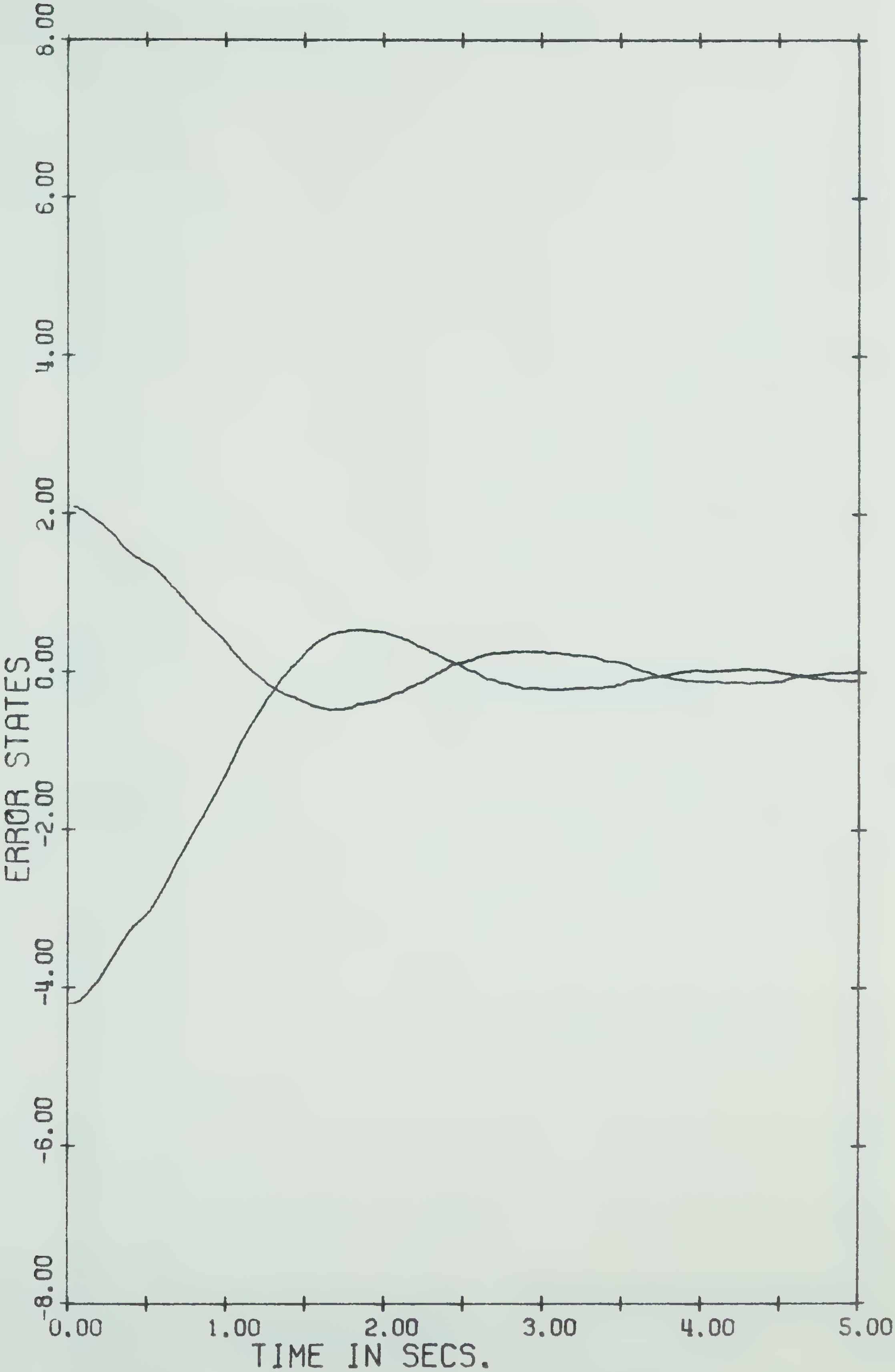

FIGURE 3.16: Position deviations with predictor: τ=0.50 second, τ_p=0.30 second. Cost J_{E1}.

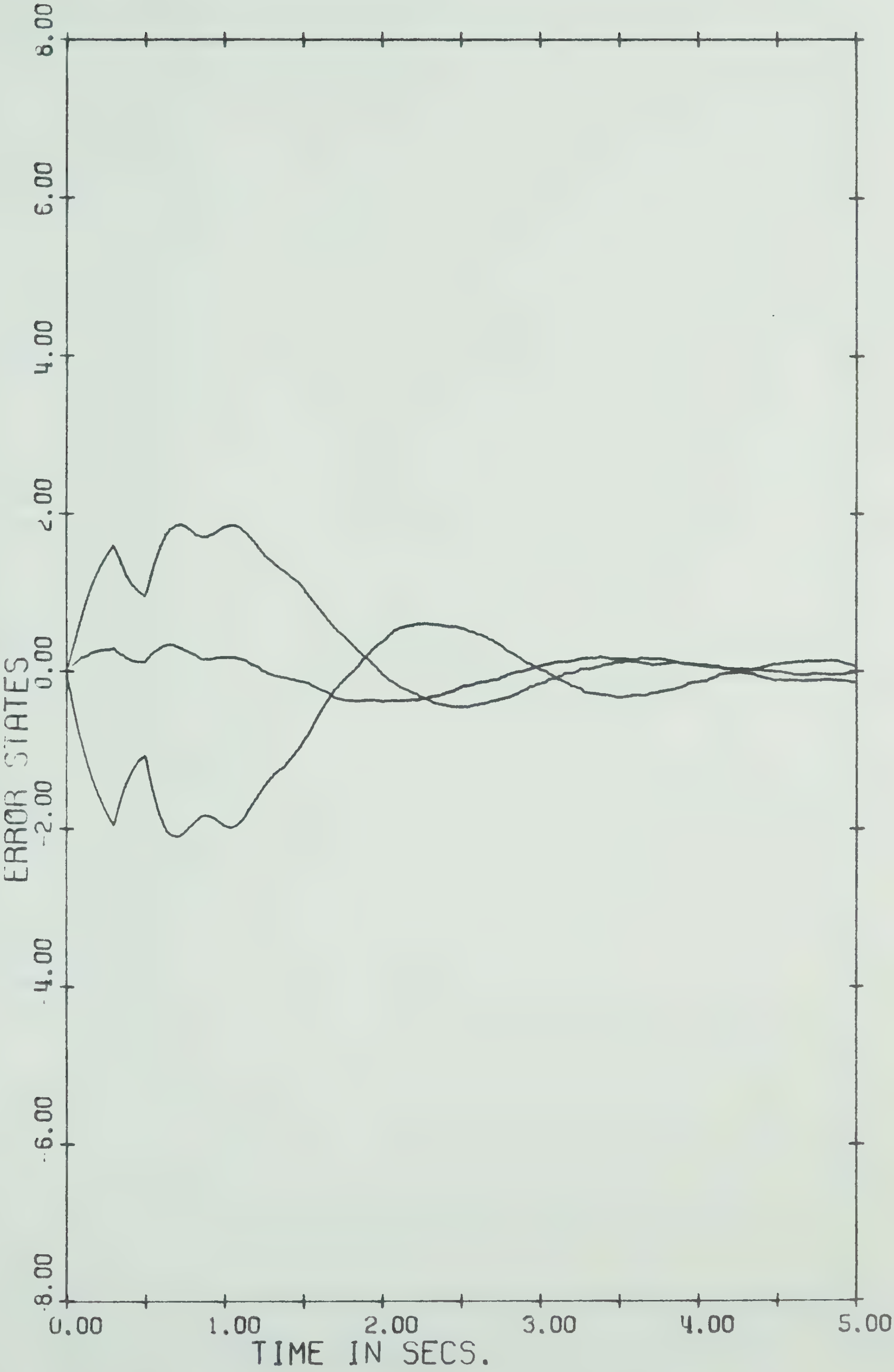

FIGURE 3.17: Velocity deviations with predictor: $\tau=0.50$, $\tau_p=0.30$. Cost J_{E1}.

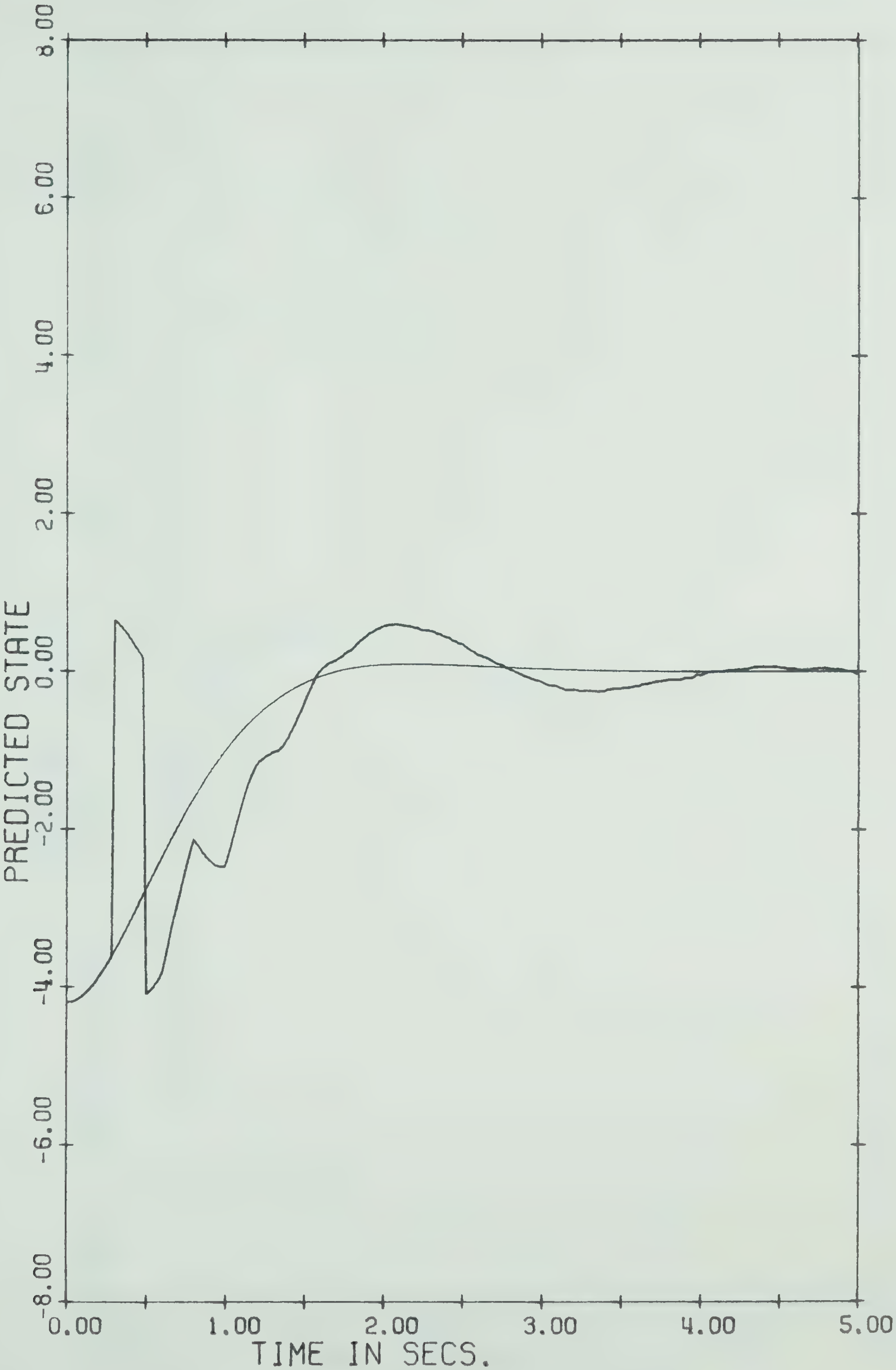

FIGURE 3.18: Predicted state of δw_1 with predictor: τ=0.50, τ_p=0.30. Cost J_{E1}.

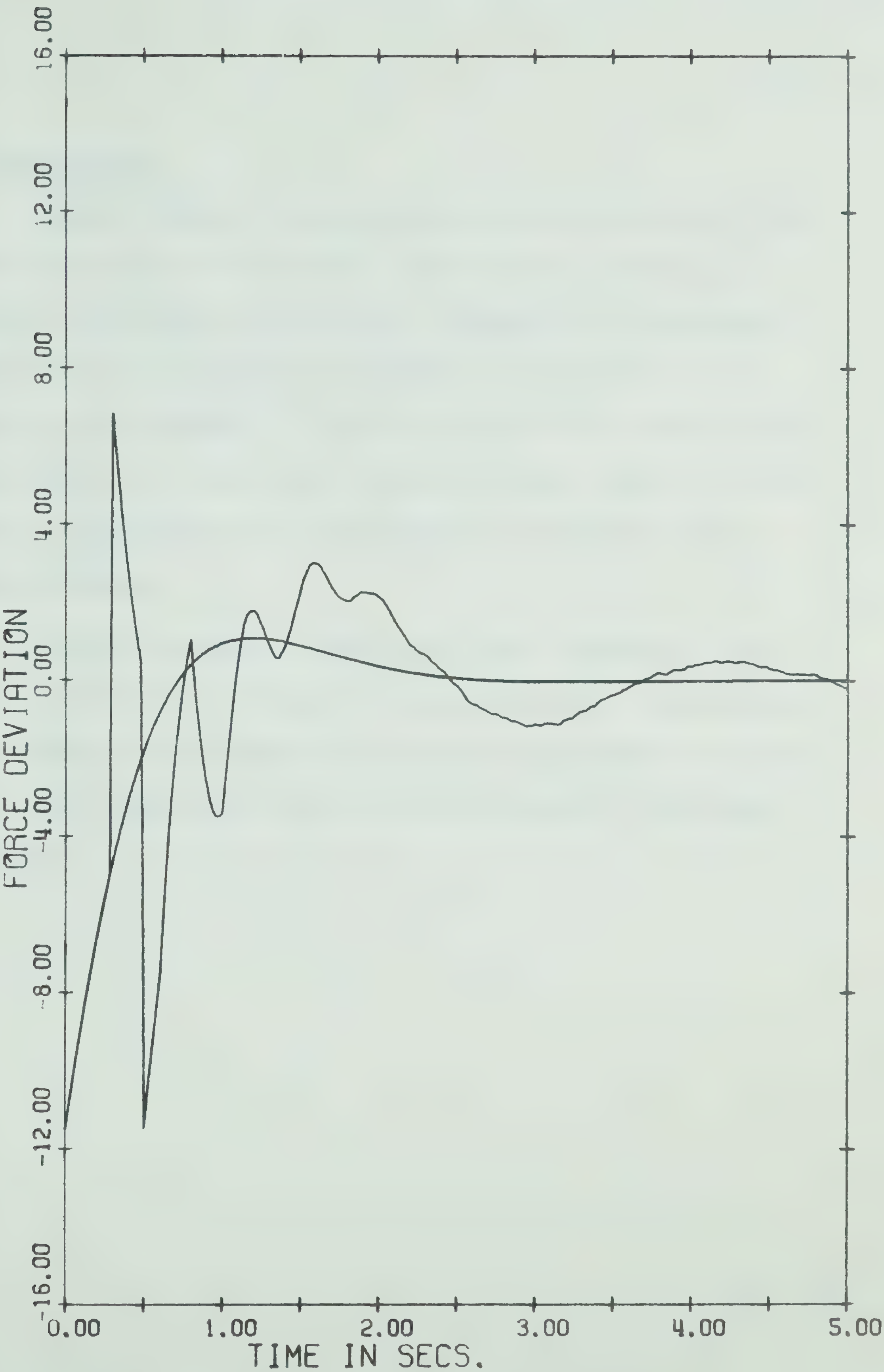

FIGURE 3.19: Control for second vehicle (δF_2) with predictor: $\tau=0.50$, $\tau_p=0.30$. Cost J_{E1}

3.4 Conclusions

Whereas regulator response may be quite acceptable for short time delays (about 0.1 second or less) without a predictor, it rapidly deteriorates as the delay is increased and soon becomes unstable. The exact point of instability is dependent on the type of system employed. It has been shown in this chapter that the inclusion of a least mean-squared predictor ensures acceptable vehicle performance even in the presence of relatively large feedback delays.

Inexact knowledge of system delay time is,however, seen to be quite detrimental to overall system performance. Nevertheless, considering that the delay time can be estimated quite accurately, this latter effect need not be of serious concern to the system designer.

CHAPTER FOUR

OPTIMAL STEADY-STATE CONTROL OF VEHICLES IN THE PRESENCE OF MEASUREMENT NOISE DRIVING NOISE AND FEEDBACK TIME DELAY

ABSTRACT

For the system having both plant and measurement noise as well as feedback time delay, it is shown that the optimal control is generated by the cascade combination of a Kalman filter and a least mean-squared predictor. The results of simulating the optimally controlled vehicle system with plant and measurement noise of unit variance together with a feedback time delay of 0.3 second are given. The observed response of the optimal system when the filter and predictor have no knowledge of the initial vehicle states is also presented.

4.1 The optimal system

In chapter three it was shown that (see equations (3-14) and (3-15)) the optimal control when the system has plant disturbances and feedback time delay is given by

$$\underline{u}^*(t) = -L^* \, \hat{\underline{x}}_p(t) = -L^*\{\underline{y}_D(t+\tau) + \ell^{A\tau} \underline{y}_S(t)\} \tag{4-1}$$

where

$$\underline{y}_S(t) = \int_{t_o}^{t-\tau} \Phi(t-\tau,\sigma) \, \underline{w}(\sigma) \, d\sigma \; ; \; t-\tau \geq t_o \; .$$

To distinguish the least mean-squared predictor estimate of $\underline{x}(t)$ of chapter three from the Kalman filter estimate of $\underline{x}(t)$ of chapter two, the notation $\hat{\underline{x}}_p$ is now used in referring to the former. The latter will subsequently be referred to as $\hat{\underline{x}}_K$. If a measurement disturbance term is now appended to equation (3-1b) such that

$$\underline{y}(t) = \underline{x}(t-\tau) + \underline{v}(t-\tau) \tag{4-2}$$

then $\underline{y}_S(t)$ becomes

$$\underline{y}_S(t) = \int_{t_o}^{t-\tau} \Phi(t-\tau,\sigma)\, \underline{w}(\sigma)\, d\sigma + \underline{v}(t-\tau);\; t-\tau \geq t_o$$

$$= \underline{x}_S(t-\tau) + \underline{v}\,(t-\tau) \quad . \tag{4-3}$$

Reduction of the stochastic effects of the control effort $\underline{u}(t)$ in equation (4-1) thus requires that $\underline{y}_S(t)$, the stochastic disturbance term, be made as small as possible. In chapter two it was shown that, with no time delay,

$$\hat{\underline{x}}_K\,(t) = \underline{x}(t) + \underline{\delta}_N(t)$$

where $\hat{\underline{x}}_K(t)$ is the Kalman filter estimate of the state $\underline{x}(t)$, and $\underline{\delta}_N(t)$ is the estimation error. It must be noted that $||\underline{\delta}_N(t)|| << ||\underline{v}(t)||$ if the filter works properly. If the input to the predictor is taken as

$$\underline{y}_K(t) = \hat{\underline{x}}_K(t-\tau) = \underline{x}(t-\tau) + \underline{\delta}_N(t-\tau), \tag{4-4}$$

then we have

$$\underline{y}_{KS}(t) = \underline{x}_S(t-\tau) + \underline{\delta}_N(t-\tau) \tag{4-5}$$

and $||\underline{y}_{KS}(t)||<<||\underline{y}_S(t)||$ (Compare equation (4-5) with equation (4-3).

Thus, it can be intuitively concluded that, for the case where the system is subject to both plant and measurement noise as well as feedback time delay, the optimal regulator should include both the Kalman filter and the least mean-squared predictor (LMSP). A rigorous proof of this conclusion has been provided by Kleinman [23]. A schematic representation of this optimal stochastic regulator with estimator and predictor (OSREP) is shown in figure 4.1.

4.2 Simulation results

Two cases are treated here.

Case 1: Plant, filter, and predictor have identical initial states - A comparison of the performance of the deterministic optimal system of figure 2.1 with feedback time delay (here called the NOSRD), with that of the optimal system of figure 4.1 can be made by comparing tables 4-1 and 4-2. The mean-square deviation (MSD) recorded in these tables are as defined in chapter two. Figures 4.2, 4.3, and 4.4 give the response of the noisy system with time delay when neither predictor nor filter is present.[25]

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[25] The simulated vehicle queue dynamics assume unity noise variance for both measurement and plant noise while the feedback delay is taken to be 0.3 second.

Except for small random perturbations, when both the predictor and filter are included the response of the system is essentially that of the deterministic model of appendix one and hence is not repeated here.

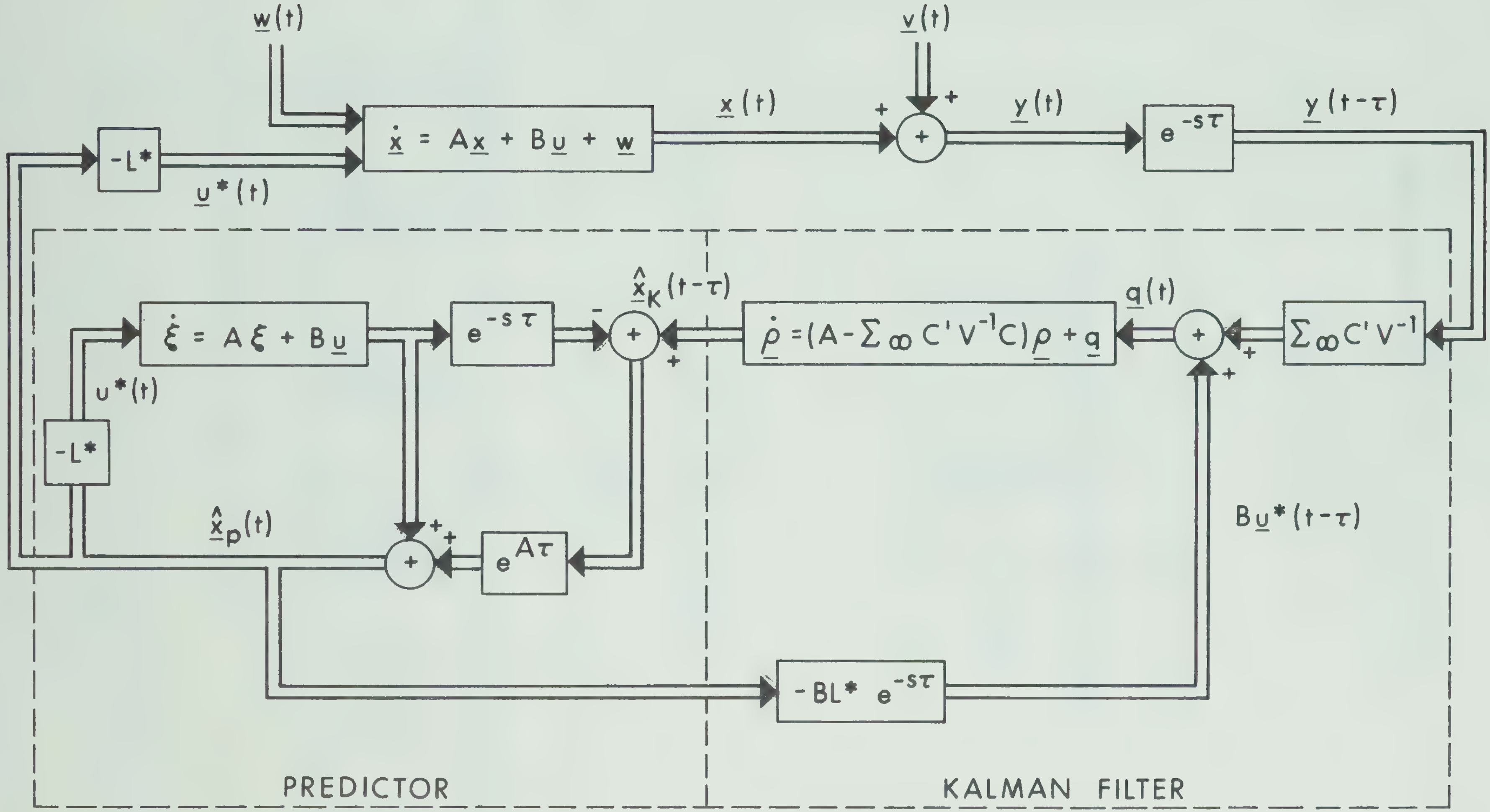

FIGURE 4.1: Optimal regulator system in the presence of plant and measurement noise as well as feedback time delay (OSREP)

TABLE 4-1

Mean-square deviations with the NOSRD:

$\sigma_N^2=1$, $\tau=0.3$ second

x	y	u
0.0797 0.0535 0.0717 0.0242 0.0108	1.0707 0.9045 1.0999 0.9925 0.8960	8.7118 11.0984 8.8251

TABLE 4-2

Mean-square deviations with the OSREP:

$\sigma_N^2=1$, $\tau=0.3$ second

x	y	$\hat{x}_K$	$\hat{x}_P$	u	$\hat{x}_P$ †
0.0017 0.0110 0.0055 0.0032 0.0010	0.8436 0.8022 0.9511 0.9265 0.8642	0.0007 0.0026 0.0014 0.0041 0.0017	0.0009 0.0021 0.0009 0.00036 0.0010	0.0111 0.0158 0.0175	0.0014 0.0151 0.0042 0.0036 0.0018

† Comments in the footnote of table 2-4 apply.

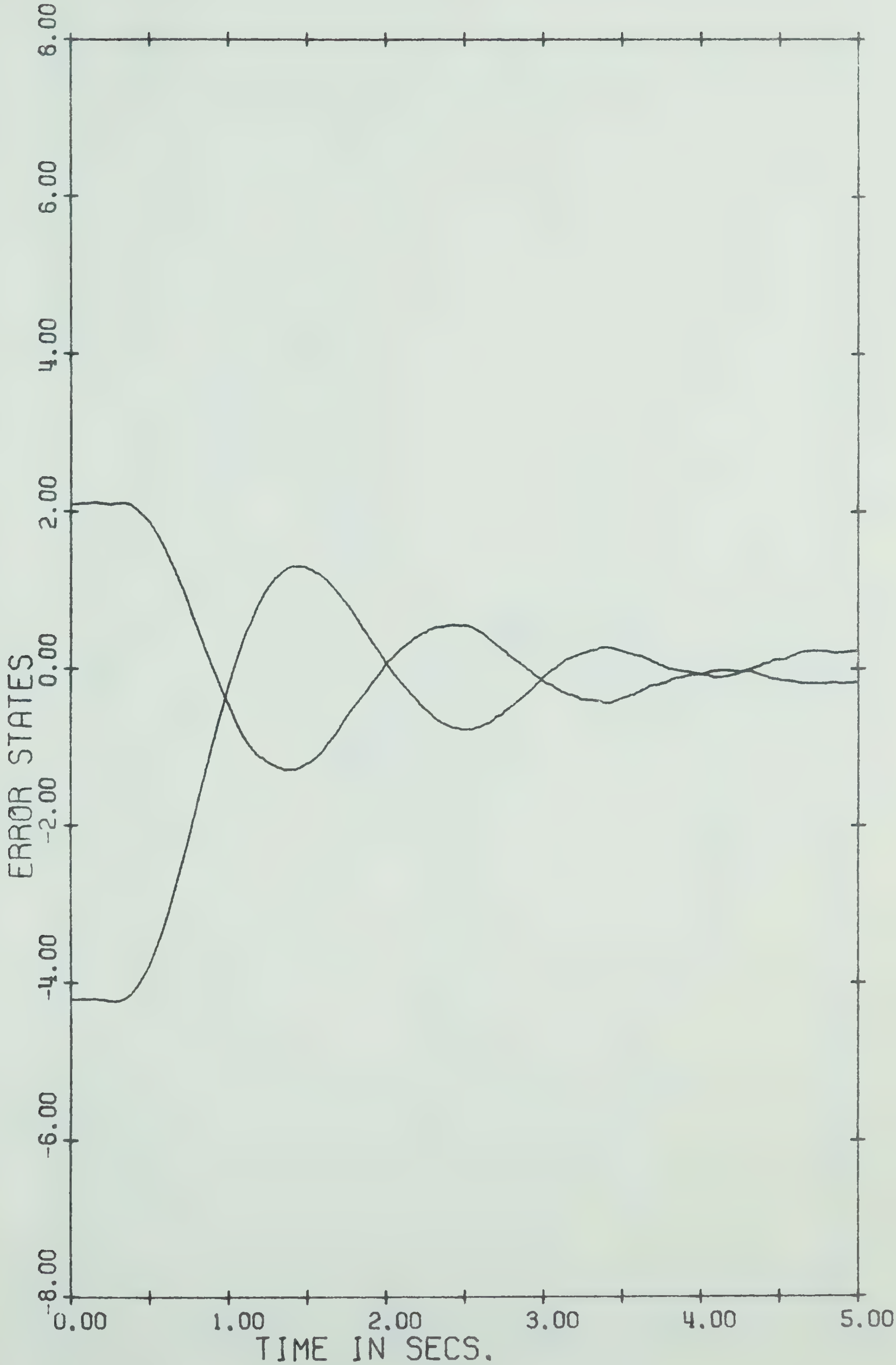

FIGURE 4.2: Position deviations with the NOSRD: $\sigma_N^2=1$, $\tau=0.3$ second. Cost J_{E1}.

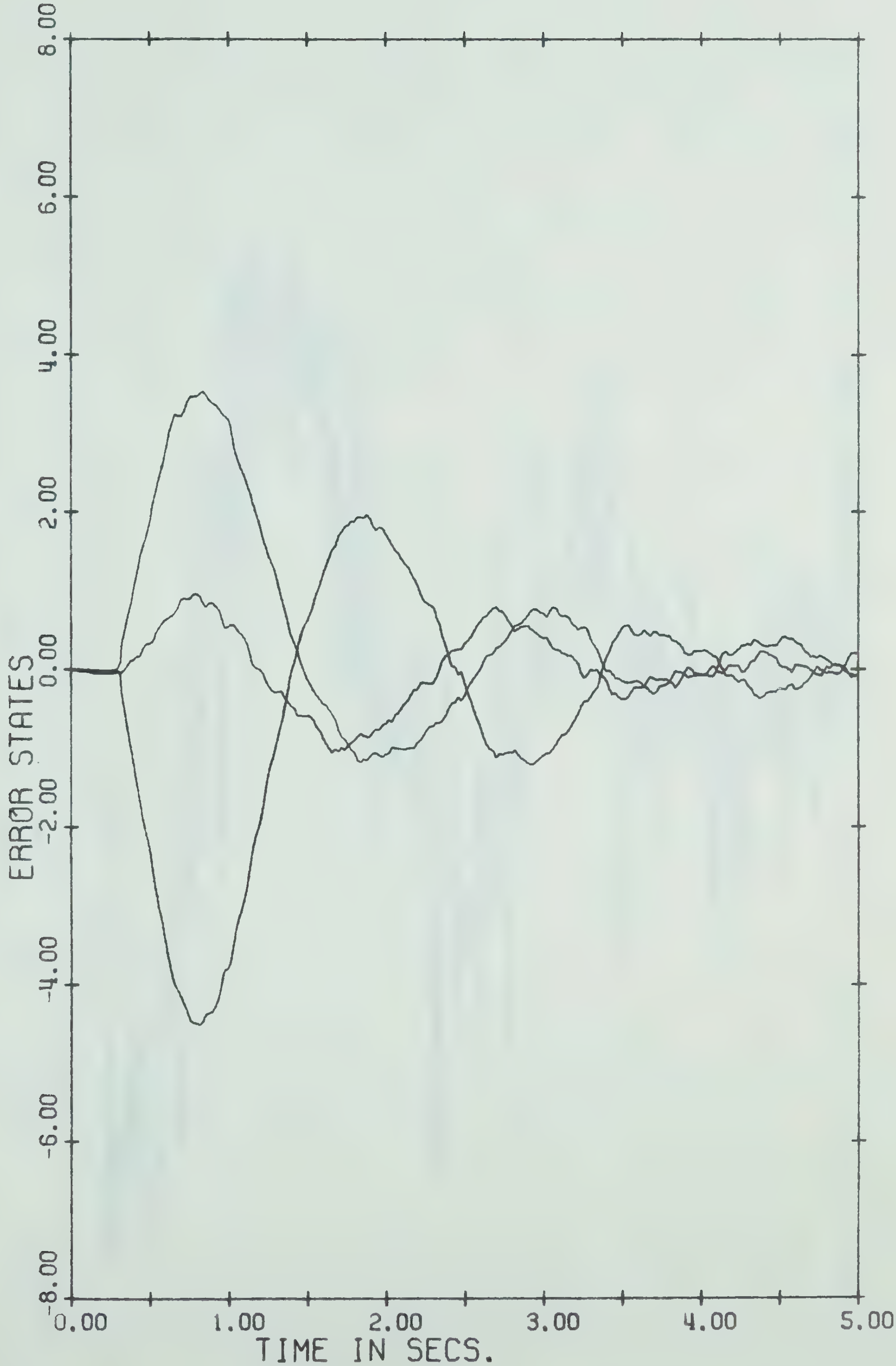

FIGURE 4.3: Velocity deviations with the NOSRD: $\sigma_N^2=1$, $\tau=0.3$ second. Cost J_{E1}.

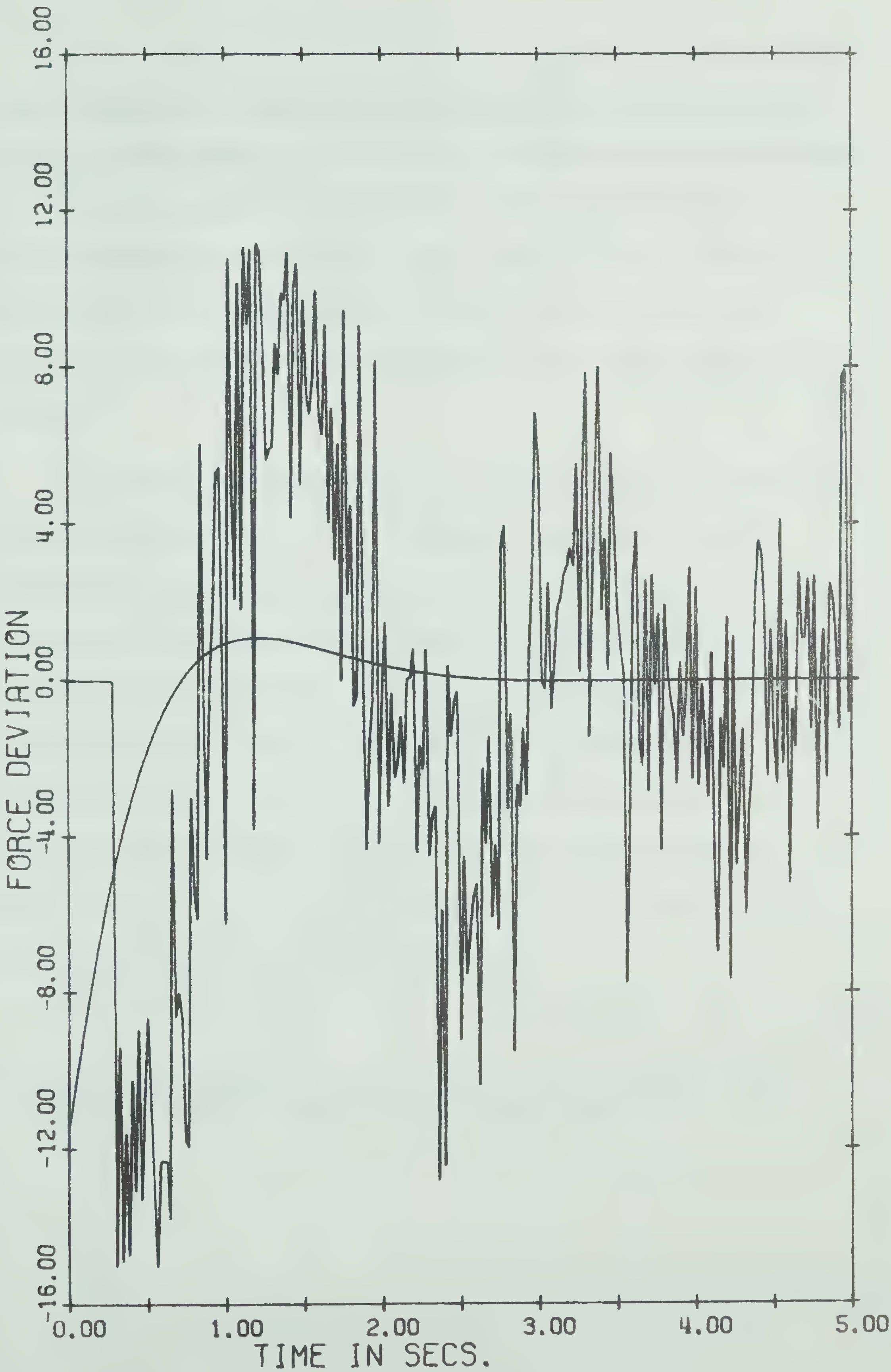

FIGURE 4.4: Control for second vehicle (δF_2) with the NOSRD: $\sigma_N^2=1$, $\tau=0.3$ second. Cost J_{E1}.

<u>Case 2: Plant has different initial conditions from those of the predictor and estimator</u> - Figure 4-5 to figure 4-8 give the response of the optimal system when the filter and predictor do not have exact knowledge of the initial plant states.[26] After some time has elapsed, it is seen that the filter locks on to the plant states, and the optimal system performs in the usual fashion from then on.

That the optimal system with the Kalman filter and least mean-squared predictor gives better regulator performance than the deterministic regulator of appendix one (in the presence of stochastic disturbances and feedback time delay) is certainly beyond dispute. Compared with the performance of the optimal deterministic regulator of appendix one in the presence of plant noise, measurement noise, and feedback time delay, it is moreover obvious that the optimal system of figure 4.1 gives not only better regulation, but what is also important, better regulation is achieved with vastly reduced demands on the vehicle power source.

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[26] Plant initial states = $\underline{x} = [0 -4.2\ 0\ 2.1\ 0]^1$; filter initial states = predictor initial states = null vector.

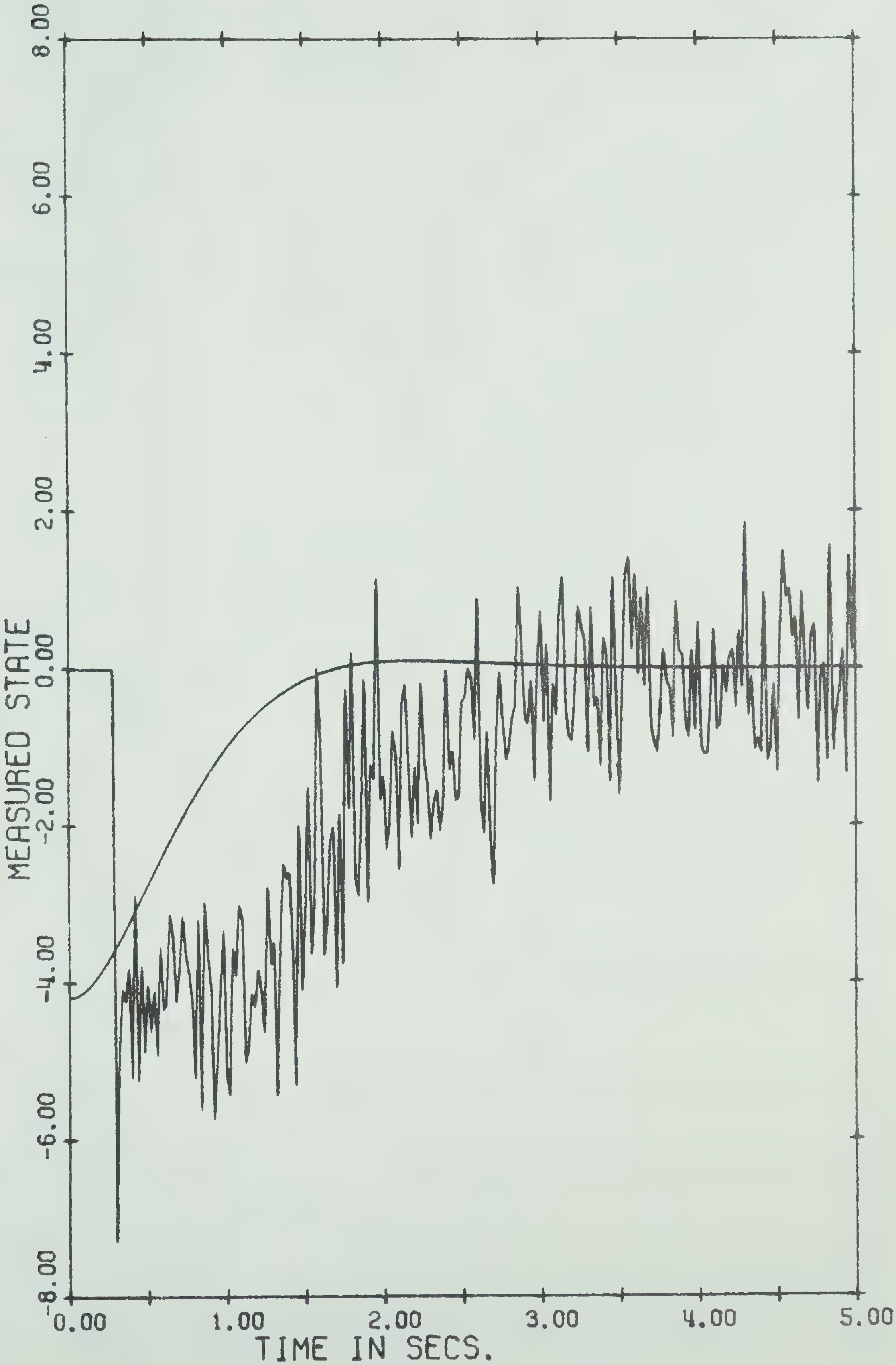

FIGURE 4.5: Measured state of $\delta w_1 : \sigma_N^2 = 1$, $\tau = 0.3$ second.
$\underline{x}(\underline{0}) = [0\ -4.2\ 0\ 2.1\ 0]'$, $\hat{\underline{x}}_p(\underline{0}) = \hat{x}_k(\underline{0}) = \underline{0}$. Cost J_{E1}.

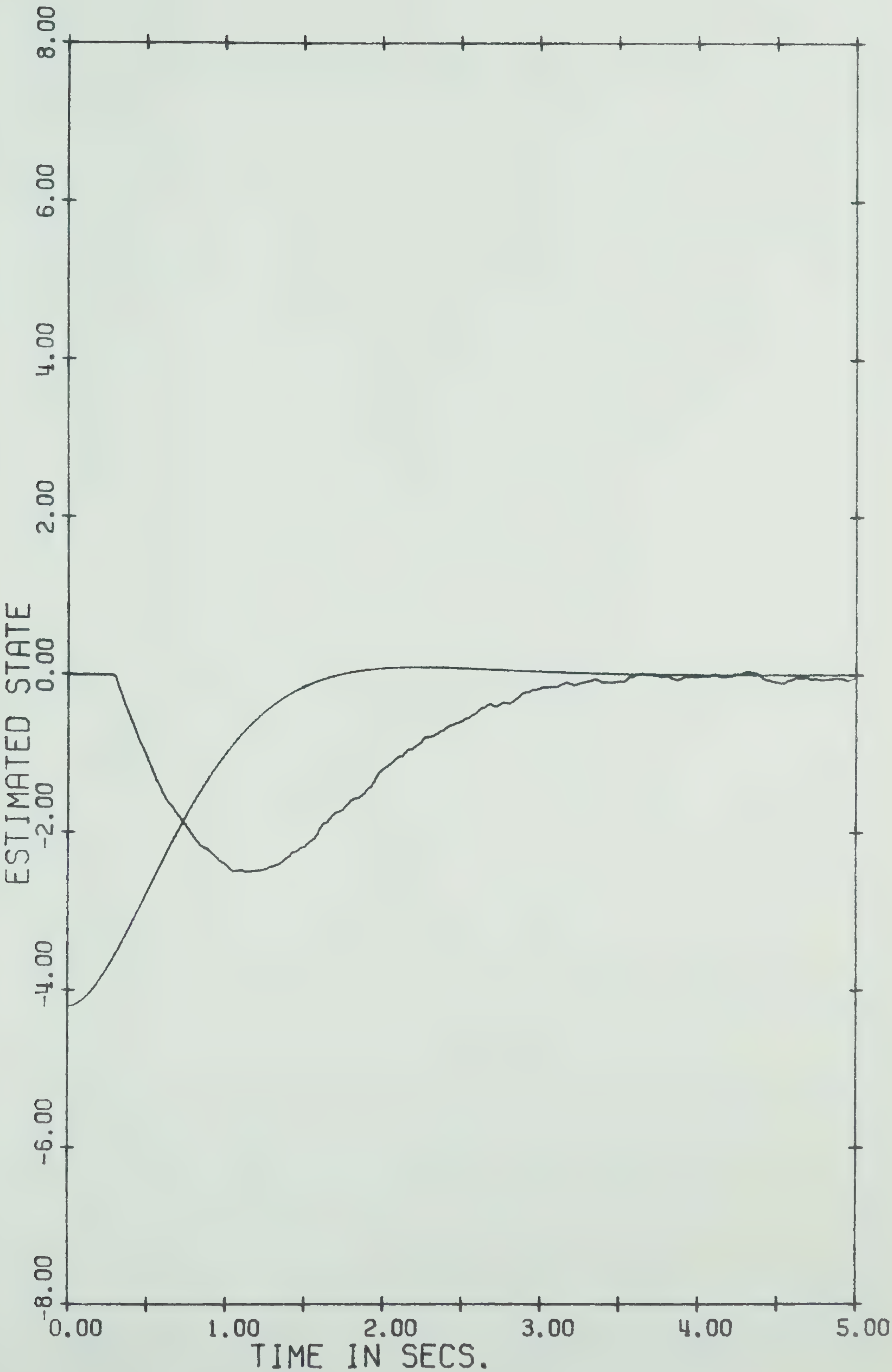

FIGURE 4.6: Estimated state of δw_1 : $\sigma_N^2=1$, $\tau=0.3$ second. $\underline{x}(\underline{0})=[0\ -4.2\ 0\ 2.1\ 0]'$, $\hat{\underline{x}}_p(\underline{0})=\hat{\underline{x}}(\underline{0})=\underline{0}$. Cost J_{E1}

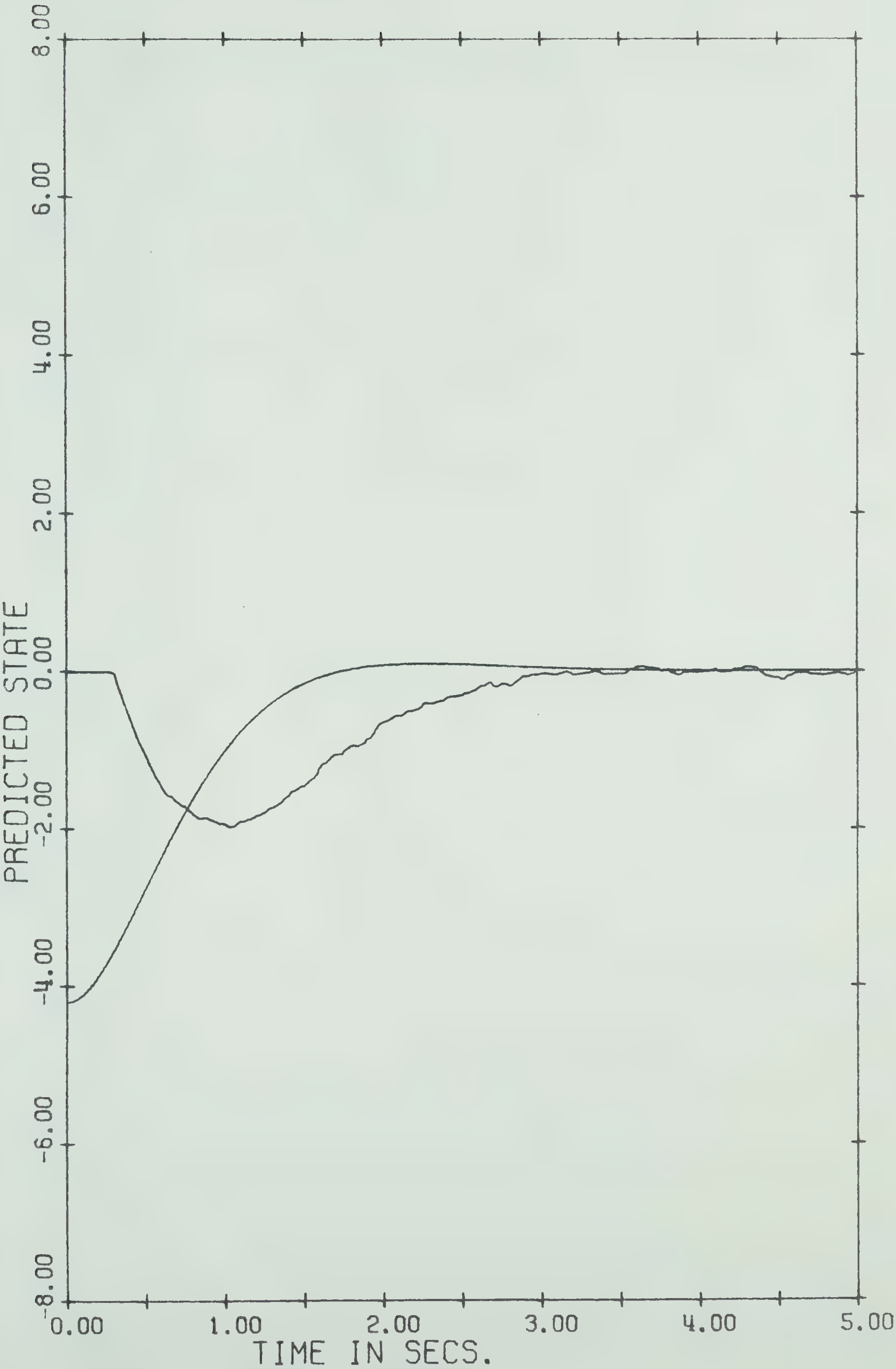

FIGURE 4.7: Predicted state of δw_1 : $\sigma_N^2=1.0$, $\tau=0.3$ second. $\underline{x}(\underline{0})=[0\ -4.2\ 0\ 2.1\ 0]'$, $\underline{\hat{x}}_P(\underline{0})=\underline{\hat{x}}_K(\underline{0})=\underline{0}$. Cost J_{E1}.

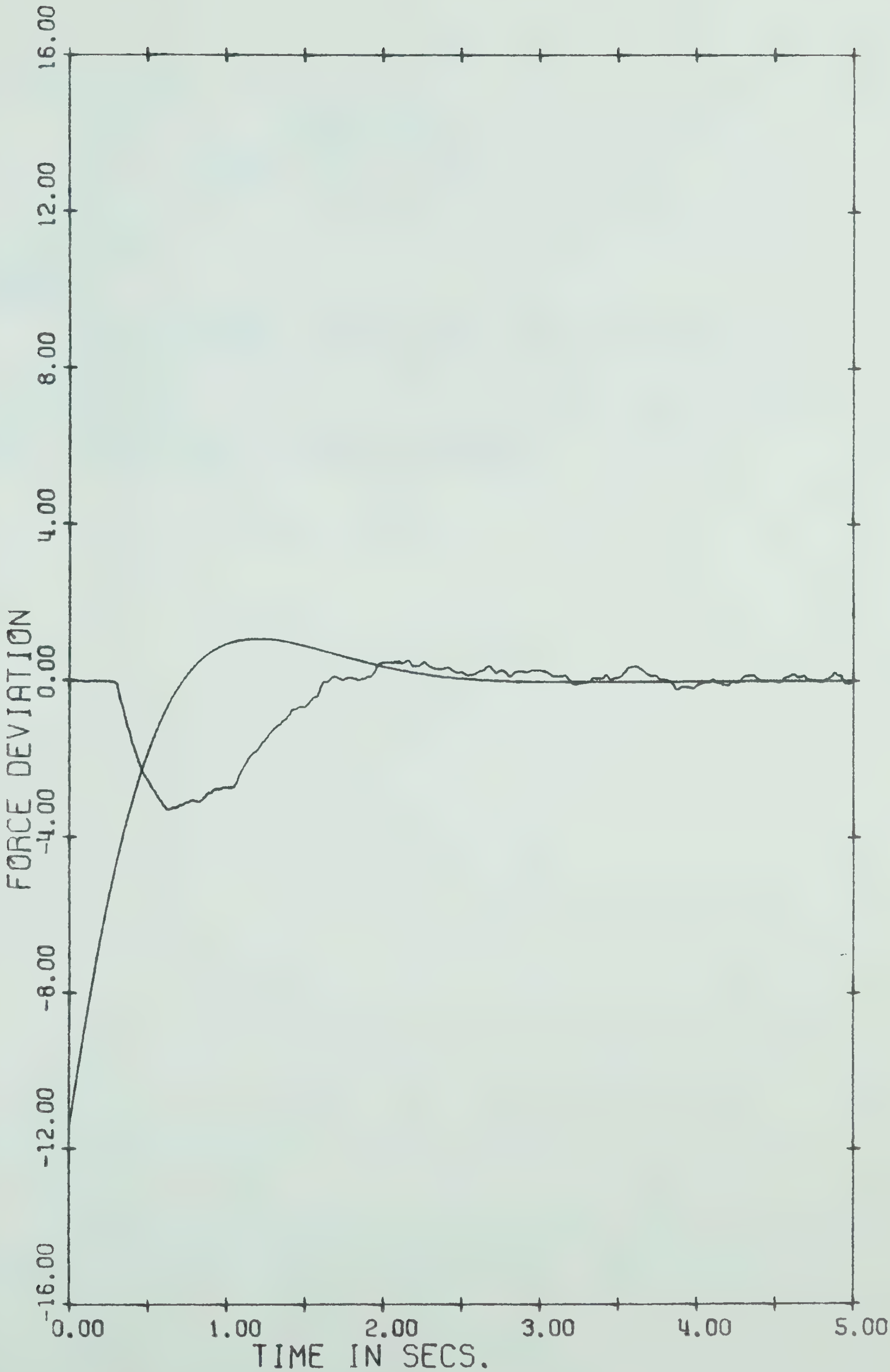

FIGURE 4.8: Control for second vehicle (δF_2): $\sigma_N^2 = 1.0$, τ=0.3 second. $\underline{x}(o)$ [0-4.2 0 2.1 0]′ $\hat{\underline{x}}_p(\underline{0})=\hat{\underline{x}}_k(\underline{0})=\underline{0}$. Cost J_{E1}

CHAPTER FIVE

SUMMARY AND CONCLUSIONS

ABSTRACT

Several concluding remarks based on the work reported in the preceding four chapters are presented. Some suggestions for future research possibilities are advanced.

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[illegible]

[illegible]

5.1 Conclusions

The work reported in the preceding four chapters of this thesis can be summarized as follows. It has been shown that an optimal stochastic controller can be designed for controlling a string of vehicles when random vehicle disturbances, state measurement noise, and feedback time delay are present. This controller, when implemented in the real world, will perform in a manner superior to that of the previously considered optimal deterministic regulator, which is optimal only when noise and time delay are absent.

The severity with which plant noise, measurement noise, and time delay will affect vehicle performance in the optimal automatic system will, in large measure, ultimately depend upon the physical characteristics of the system employed. To name just a few : the communication network, system operating speed, vehicle power plant size, number of vehicles controlled, vehicle mass, will all be deciding factors in determining the control strategy. It is to be expected then that present studies can only concern themselves with general aspects of vehicle regulation with the hope of developing the means to obtain the best possible performance commensurate with economic and technical realities.

Past work has shown that good regulatory systems can be designed by appropriate use of optimal control theory by assuming

that the physical components of the automatic system are capable of meeting the requirements of the controller. Since the power plant size will in general determine the accelerating capability of any given vehicle, many control laws will most certainly be limited by the need for a prohibitively large power source (with its obvious attendant impact on operating costs and passenger comfort). Because of its resultant effects on vehicle response, regulator power requirements cannot, moreover, be reduced by simply increasing the penalty associated with the expenditure of input control energy. An increase in the relative importance of expended energy would certainly reduce power plant size, but this may also have the attendant undesirable feature of forcing the system to fail in meeting its performance specifications. What has been shown in this thesis, however, is that by proper overall system design, not only can power plant size and passenger discomfort be significantly reduced, but, what is even more important, regulation as well can be enhanced.

It has also been shown that the susceptibility of the control system to the corrupting influence of feedback time delay will hinge heavily upon the feedback control strategy that is used. Given that it is economically desirable to increase the number of vehicles which a single control center can handle (resulting in increased computation time for the required vehicle controls) it

is evident that a scheme for reducing the undesirable effects of the delay is certainly attractive. Moreover, since it is not certain how severe the effects of feedback time delay will be on final system performance, nor how large the actual delay will be, the inclusion of a predictor in the designed system may be more necessary than previously thought.

5.2 Suggestions for future research

A major concern in the area of optimal regulator design is attributable to an inability to specify, systematically, the performance criteria, by choice of the weighting matrices Q and R, and the noise covariance matrices W(t) and V(t) which will result in a desirable system response. Further research is required here to make the optimal design approach more attractive and less expensive.

The plant noise considered in this thesis has served as a model for actuator noise and model uncertainties. Since modeling errors are certainly not white, it would be useful to study the effects of the introduction of colored noise into the design and performance of an optimal controller for a string of vehicles.

Design of an optimal controller for use under emergency conditions is, undoubtedly, of grave concern.

The effects of model nonlinearities on system performance and design are also worth considering.

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APPENDIX ONE

THE DETERMINISTIC STEADY-STATE VEHICLE REGULATOR

Most of the research done on the position and velocity regulator has followed quite closely the original format proposed by Athans and Levine [27].

Modeling each vehicle in a string as a second-order dynamical system with non-linear damping, the equations of motion are written in the general form [30][A1]

$$\frac{d}{dt} Z_k(t) = y_k(t) \tag{A1-1a}$$

$$m_k \frac{d}{dt} y_k(t) = -g_k[y_k(t)] + f_k(t) \; ; \; k=1,2\ldots,N \tag{A1-1b}$$

where $Z_k(t)$ and $y_k(t)$ are the position and velocity of the k^{th} vehicle, respectively, and $f_k(t)$ is the force applied to the k^{th} vehicle at time t. The mass of the k^{th} vehicle is given by m_k while N is the total number of vehicles in the string.[A2]

[A1] Motion is assumed to proceed along a flat straight guideway.

[A2] Figure A1-1 depicts a three vehicle string.

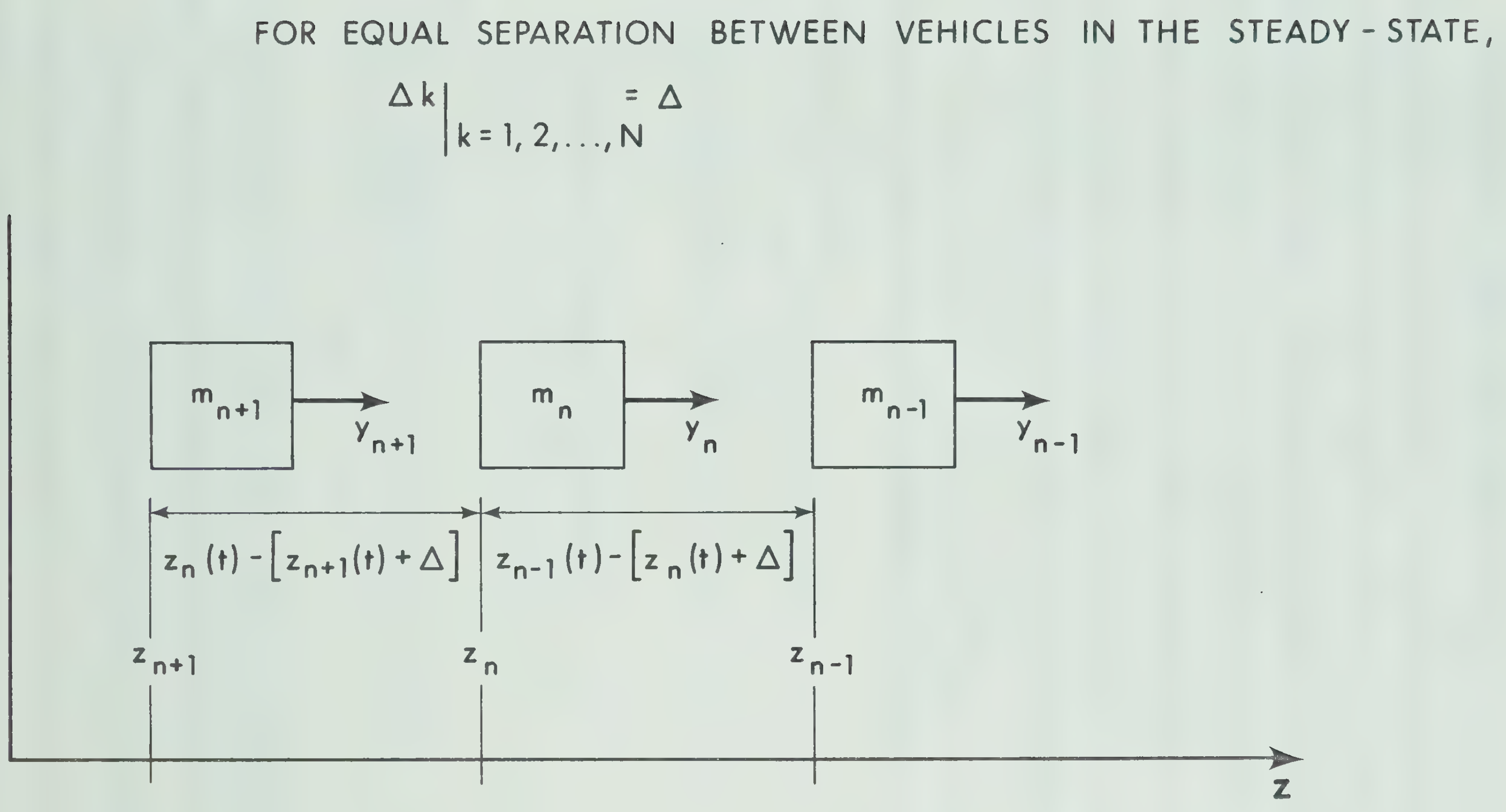

FIGURE A1.1 : Three vehicles moving in a string.

To write the equations of motion (A1-1) in terms of error state variables one then defines: 1. the deviations from a desired separation distance between adjacent vehicles Δ_k as the position state variable, and 2. the velocity deviations from the prescribed mean string velocity V_o as the velocity state variable. Defining the state variables in this way forces the equations of motion of adjacent vehicles to become coupled and also allows linearization of the non-linear damping term about the mean string velocity V_o.

The result is a set of linearized differential equations for the position and velocity error variables, $\delta w_k(t)$ and $\delta y_k(t)$, respectively, given by

$$\frac{d}{dt}\,\delta w_k(t) = \delta y_k(t) - \delta y_{k+1}(t) \tag{A1-2a}$$

$$\frac{d}{dt}\,\delta y_k(t) = \frac{-\alpha k}{m_k}\,\delta y_k(t) + \frac{1}{m_k}\,\delta f_k(t) \tag{A1-2b}$$

where

$$\alpha_k = \left.\frac{\partial g_k[y_k(t)]}{\partial y_k(t)}\right|_{y_k(t) = V_o}$$

is the first order term in the Taylor series expansion of the non-linear drag term, $g_k[y_k(t)]$, about the mean string velocity V_o.

Interlacing the velocity deviations with the position deviations, the linearized state-output equations for an N-vehicle string can be written in the form

$$\frac{d}{dt}\underline{x}(t) = A\,\underline{x}(t) + B\,\underline{u}(t)\ ;\ \underline{x}(t_0) = \underline{x}_0 \tag{A1-3a}$$

$$\underline{y}(t) = C\,\underline{x}(t) \tag{A1-3b}$$

where $\underline{x}(t)$ is the (2N-1) - dimensional state vector, $\underline{u}(t)$ is the N-dimensional control vector, $\underline{y}(t)$ is the r-dimensional observation vector, and A, B, C are (2N-1) x (2N-1), (2N-1) x N, and rx(2N-1) matrices,respectively.[A3]

Specification of a quadratic cost functional of the type

$$J = \frac{1}{2}\int_0^\infty [\underline{x}'(t)Q\underline{x}(t) + \underline{u}'(t)R\underline{u}(t)]\,dt \tag{A1-4}$$

where R and Q are positive definite matrices then reduces the problem to the standard linear regulator problem of optimal control theory [5, 24] .[A4]

The control which minimizes J for any set of initial conditions

---

[A3] Note that in the vehicle regulator problem, the velocity and position deviations of every vehicle are measured. Hence, the dimension of the measurement vector, r, is the same as that of $\underline{x}(t)$, and the matrix C is the (2N-1)x(2N-1) identity matrix.

[A4] A quadratic cost is specified for three main reasons:1. computational convenience, 2. equal penalization of positive and negative deviations in the state variables, and 3. penalization of large deviations relatively more severely than small ones.
Since the quadratic cost does not provide for an infinite penalty when the vehicles touch,however, it is only valid under normal operating conditions[4].

on the state vector $\underline{x}(t)$ is given by

$$\underline{u}^*(t) = - R^{-1}B'\hat{K}\,\underline{x}(t) = - L^*\,\underline{x}(t) \tag{A1-5}$$

$\hat{K}$ is the real symmetric positive definite constant matrix which can be found either by solving : 1. the non-linear algebraic equation

$$-\hat{K} A - A'\hat{K} + \hat{K} BR^{-1} B'\hat{K} - Q = 0, \tag{A1-6}$$

or 2. the matrix differential equation

$$\frac{d}{d\tau} K(\tau) = K(\tau)A + A'K(\tau) - K(\tau)BR^{-1}B'K(\tau) + Q \tag{A1-7}$$

with the initial condition $K(0) = 0$, and then setting

$$\hat{K} = \lim_{\tau\to\infty} K(\tau)$$ [A5]

Consider now the case of three vehicles where there are two position deviations and three velocity deviations. Assuming that $m_1=m_2=m_3=1$ and $\alpha_1=\alpha_2=\alpha_3=1$, the state equations can be written as[A6]

[A5] The assumptions of controllability and no terminal cost imply that the limit $K(\tau)$ exists, is unique, and is $\hat{K}$: that is [5]

$$\lim_{\tau\to\infty} K(\tau) = \hat{K}$$

where $\hat{K}$ is the positive definite matrix which is the solution of (A1-6)

[A6] Note that choosing $m_1=m_2=m_3=1$ in no way restricts the validity of the observed results. If $m_1\neq m_2\neq m_3\neq 1$, then some other choice for the relative values of the weighting matrices Q and R of equation (A1-4) can be found such that the response is identical to that when all the vehicle masses are equal to 1.

$$\frac{d}{dt}\begin{bmatrix}\delta y_1(t)\\ \delta w_1(t)\\ \delta y_2(t)\\ \delta w_2(t)\\ \delta y_3(t)\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0 & 0 & 0\\ 1 & 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0 & -1\end{bmatrix}\begin{bmatrix}\delta y_1(t)\\ \delta w_1(t)\\ \delta y_2(t)\\ \delta w_2(t)\\ \delta y_3(t)\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\delta f_1(t)\\ \delta f_2(t)\\ \delta f_3(t)\end{bmatrix} \quad \text{(A1-8)}$$

Choosing the cost function

$$J = \frac{1}{2}\int_0^\infty [10(\delta w_1^2(t)+\delta w_2^2(t)) + \delta f_1^2(t)+\delta f_2^2(t)+\delta f_3^2(t)]dt \quad \text{(A1-9)}$$

then allows calculation of $\hat{K}$.

$$\hat{K}=\begin{bmatrix}1.263 & 2.494 & -0.819 & 0.668 & -0.444\\ 2.494 & 7.434 & -1.826 & 1.123 & -0.668\\ -0.819 & -1.826 & 1.638 & 1.826 & -0.819\\ 0.668 & 1.123 & 1.826 & 7.434 & -2.494\\ -0.444 & -0.668 & -0.819 & -2.494 & 1.263\end{bmatrix} \quad \text{(A1-10)}$$

The optimal feedback controls are, from equation (A1-5),

$$\delta f_1(t)=-[1.263\delta y_1(t)+2.494\delta w_1(t)-0.819\delta y_2(t)+0.669\delta w_2(t)-0.444\delta y_3(t)]$$

$$\delta f_2(t)=-[-0.819\delta y_1(t)=1.826\delta w_1(t)+1.638\delta y_2(t)+2.826\delta w_2(t)-0.819\delta y_3(t)]$$

$$\delta f_3(t)=-[-0.444\delta y_1(t)-0.668\delta w_1(t)-0.819\delta y_2(t)-2.494\delta w_2(t)+1.263\delta y_3(t)]$$

In figure A1.2, the resultant optimal system is shown in block diagram form.

The deterministic steady-state vehicle regulator is thus founded on the following five basic assumptions : 1. vehicle motion is along a straight flat guideway, 2. the linearized model for the vehicle dynamics is valid, 3. no system disturbances exist, 4. no time delays are present anywhere in the system, and 5. normal vehicle operating conditions prevail.

A computer simulation of the three vehicle regulator was carried out on the IBM System/360 using CSMP. For the initial state vector [0 - 4.2 0 2.1 0]', the observed response of the system is shown in figures A1-3, A1-4, and A1-5, for the position, velocity, and force deviations, respectively.

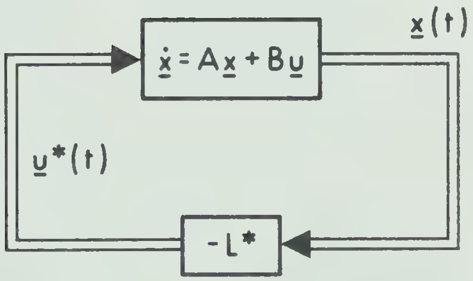

FIGURE A1-2: Optimal deterministic regulator system.

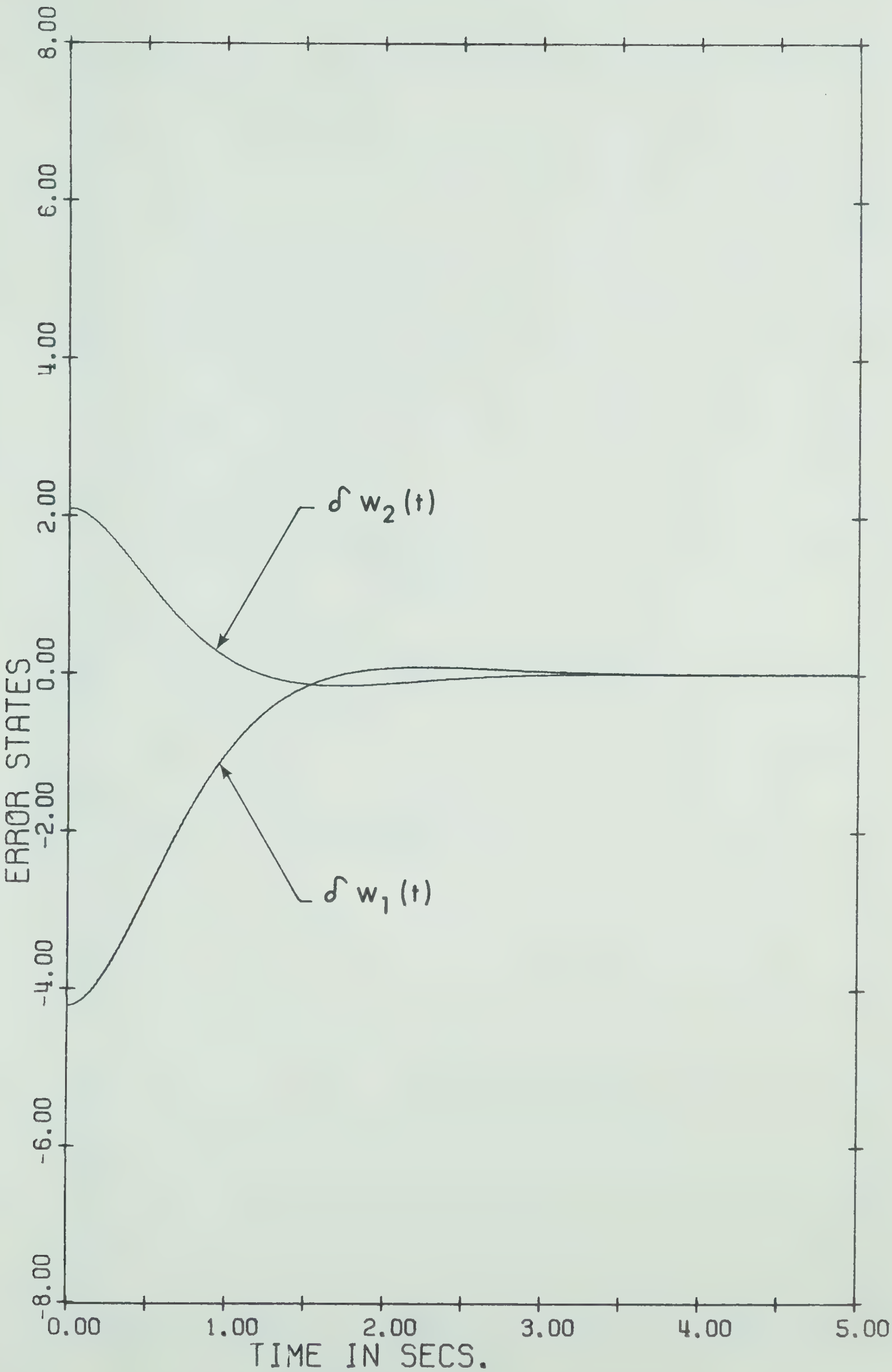

FIGURE A1.3: Position deviations for the no noise and no time delay optimal regulator

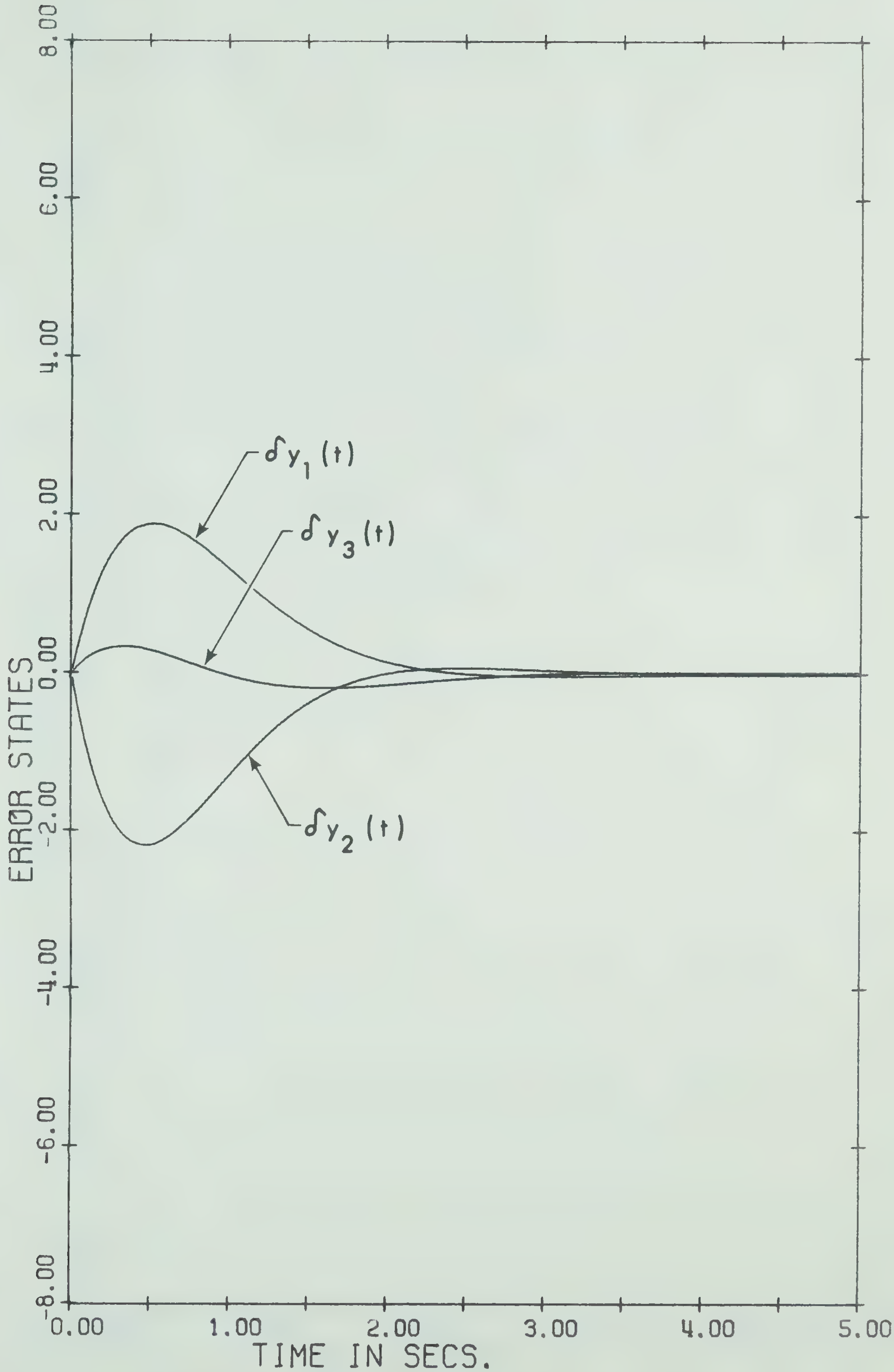

FIGURE A1.4 Velocity deviations for the no noise and no time delay optimal regulator

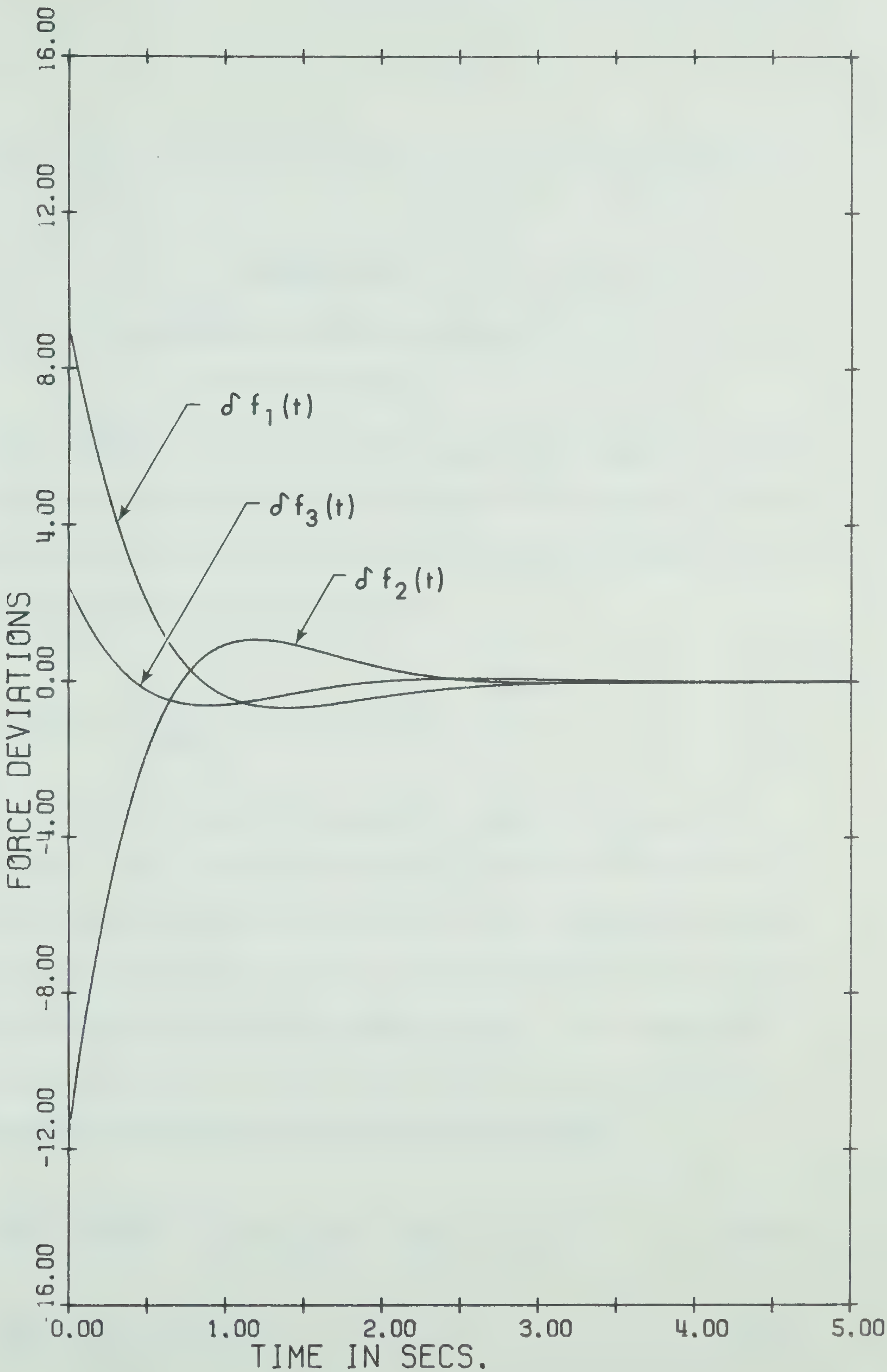

FIGURE A1.5: Corrective force deviations for the no noise and no time delay optimal regulator

APPENDIX TWO

THE CERTAINTY EQUIVALENCE PROPERTY OF OPTIMAL CONTROL

This discussion of the certainty equivalence property of optimal control is a slightly more detailed version of that given by Tse [42]. A starting point for the discussion can be easily provided by noting that minimization of the cost function

$$J_E(\underline{u}) = E\left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T (\underline{x}'Q\underline{x} + \underline{u}'\, R\, \underline{u})\ dt \right\} \tag{A2-1}$$

requires prior knowledge of the possible stochastic effects of a given control law. Knowing the stochastic effects of the control $\underline{u}(t)$, the optimal control which minimizes (A2-1) can then be chosen from the set of admissable controls. To be admissable, the controls $\underline{u}(t)$ must satisfy two important properties [42] : 1. it must be nonanticipative, and 2. it must satisfy the Lipschitz condition to guarantee the existence of $\underline{x}(t)$ and $\underline{y}(t)$ of the model

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\,\underline{u}(t) + \underline{w}(t)\ ;\ \underline{x}(\underline{0}) = \underline{x}_0 \tag{A2-2a}$$

$$\underline{y}(t) = C\,\underline{x}(t) + \underline{v}(t) \tag{A2-2b}$$

derived in section 2.1. The derivation of the certainty equivalence property now follows.

If, for the moment, the control vector is considered to be known and deterministic it is always possible to define

$$\underline{x}(t) = \underline{x}_1(t) + \underline{x}_2(t)$$

where $\underline{x}_1(t)$ and $\underline{x}_2(t)$ satisfy

$$\dot{\underline{x}}_1(t) = A\ \underline{x}_1(t) + \underline{w}(t)\ ;\ \underline{x}_1(t_o) = \underline{x}(t_o) \tag{A2-3}$$

$$\dot{\underline{x}}_2(t) = A\ \underline{x}_2(t) + B\ \underline{u}(t)\ ;\ \underline{x}_2(t_o) = \underline{0} \tag{A2-4}$$

The state $\underline{x}_2(t)$ is then completely known if $\underline{u}(t)$ is known and is given by

$$\underline{x}_2(t) = \int_{t_o}^{t} \Phi\ (t,\tau)\ B\ \underline{u}\ (\tau)\ d\tau. \tag{A2-5}$$

Its contribution to the observation $\underline{y}(t)$ can thus always be subtracted and one can define

$$\underline{y}_1(t) = \underline{y}(t) - \underline{x}_2(t) = \underline{x}_1(t) + \underline{v}(t) \tag{A2-6}$$

where $\underline{y}_1(t)$ now embodies the stochastic effects. Knowing $\underline{x}(t_0)$ then, the stochastic effects of $\underline{u}(t)$ are also known.[A1] If $\underline{u}(t)$ is not known a priori the unknown contribution of $\underline{u}(t)$ cannot be subtracted from the observation and the stochastic effects of $\underline{u}(t)$ are not evident. If the control $\underline{u}(t)$ is admissable however, one can calculate $\{\underline{u}(\tau),\ \tau\ \varepsilon[t_0,t]\}$ when $\{\underline{y}(\tau),\ \tau\ \varepsilon[t_0,t]\}$ is observed and then compute $\underline{x}_2(t)$ in (A2-5). Hence, given $\underline{x}(t_0)$ and monitoring $\{\underline{y}(\tau),\ \tau\ \varepsilon[t_0,t]\}$ the stochastic effects of the control action $\{\underline{u}(\tau),\ \tau\ \varepsilon[t_0 t]\}$ can be found.

In the more general case, the solution of (A2-2) is

$$\underline{x}(\tau) = \Phi(\tau,t)\ \underline{x}(t) + \int_t^\tau \Phi(\tau,\delta)\ B\ \underline{u}(\delta)\ d\delta + \int_t^\tau \Phi(\tau,\sigma)\ \underline{w}(\sigma)\ d\sigma\ ;\tau \geq t \qquad \text{(A2-7)}$$

Here too it can be shown that the stochastic effects of the control action $\{\underline{u}(\sigma),\sigma\ \varepsilon[t,\tau]\}$ can be deduced if $\underline{x}(t)$ is given. However, in the general case where

$$\underline{y}(t) = \underline{x}(t) + \underline{v}(t)$$

[A1] Recall that $\underline{u}(t)$ is of the form

$$\underline{u}(t) = \phi[\underline{y}(t)] = \phi[\underline{y}_1(t) + \underline{x}_2(t)].$$

Since $\underline{x}_2(t)$ is known from (A2-5) then the stochastic component of $\underline{u}(t)$ is also known.

is the observed vector, it is generally not possible to find what $\underline{x}(t)$ is. To circumvent this difficulty one can rather find the conditional stochastic effects of future control actions by treating the conditional density, $p(\underline{x}(t)/\{\underline{y}(\sigma),\ \sigma \leq t\})$, as one would the actual state $\underline{x}(t)$. The conditional density represents the sufficient statistic for describing the future stochastic effects. Therefore, it seems that a realizable control can be provided using [42]

$$\underline{u}(t) = \phi(t, p(\underline{x}(t)/\{\underline{y}(\sigma), \sigma \leq t)) \text{ for some } \phi\ (\cdot,\cdot) \tag{A2-8}$$

Because the conditional density is in the function space which is of infinite dimension, in its present form control law (A2-8) is not of much practical use. Making use of the Gaussian assumptions of section 2.1 concerning the model (A2-2) now allow parametrization of the conditional density by its conditional mean and covariance; each in a finite dimensional space. The conditional mean and covariance of the state are defined by, respectively,

$$\hat{\underline{x}}(t) \triangleq E[\underline{x}(t)/\underline{y}(\sigma),\ \sigma \leq t] \tag{A2-9}$$

$$\Sigma(t) \triangleq E[(\underline{x}(t) - \hat{\underline{x}}(t))(\underline{x}(t) - \hat{\underline{x}}(t))'/\underline{y}(\sigma),\ \sigma \leq t]. \tag{A2-10}$$

If the covariance is independent of control and observation then only the conditional mean, $\hat{\underline{x}}(t)$, is required to parametrize the

conditional density.[A2] In this case only controls of the form

$$\underline{u}(t) = \phi(t, \hat{\underline{x}}(t)) \tag{A2-11}$$

need be sought. The process being controlled is now the conditional mean process $\hat{\underline{x}}(t)$. Thus, to obtain the optimal control law in this stochastic case, one can solve an equivalent control problem where the conditional mean $\hat{\underline{x}}(t)$ is treated as the actual state of the system. This is often termed the certainty equivalence property of optimal control.

[A2] Obviously, if $\{\underline{y}(\sigma), \sigma \leq t\}$ is given, $\hat{\underline{x}}(t)$ and $\Sigma(t)$ can be calculated using equations (A2-9, A2-10) whereupon the conditional density is parametrized. If $\Sigma(t)$ is not dependent on the observed vector $\{\underline{y}(\sigma), \sigma \leq t\}$ then we can calculate $\Sigma(t)$ independently of $\{\underline{y}(\sigma), \sigma \leq t\}$ and only further seek $\hat{\underline{x}}(t)$.

APPENDIX THREE

THE STEADY-STATE KALMAN FILTER

In appendix two it is found that, as a result of the Gaussian assumptions, the conditional density can be parametrized by the conditional mean $\hat{\underline{x}}(t)$ and covariance $\Sigma(t)$. It is a relatively simple matter to show that the conditional mean defined by

$$\hat{\underline{x}} = E\{\underline{X}/\underline{Y} = \underline{y}\} = \int_{-\infty}^{\infty} \underline{x}\ f_{\underline{X}/\underline{Y}}\ (\underline{x}/\underline{y})\ \ d\underline{x} \qquad \text{(A3-1)}$$

is the best estimate of $\underline{x}(t)$; in the sense that $\hat{\underline{x}}$ is the n-vector that minimizes over all n-vectors, $\underline{z}$, the conditional expectation

$$E\{||\underline{X}-\underline{z}||^2/\underline{Y}=\underline{y}\} = E\{[\underline{X}-\underline{z}]'[\underline{X}-\underline{z}]/Y=\underline{y}\} \qquad \text{(A3-2)}$$

of the norm-squared estimation error given that $\underline{Y}$ has value $\underline{y}$. The proof of this is rather straightforward and is done in a countless number of texts [41, 42, 45]. Expand (2-19) and obtain

$$E\{||\underline{X}-\underline{z}||^2/\underline{Y}=\underline{y}\} = E\{X'X-2\underline{z}'X + \underline{z}'\underline{z}/\underline{Y}=\underline{y}\}$$

$$= E\{\underline{X}'X/\underline{Y}=\underline{y}\} - 2\underline{z}' \, E\{\underline{X}/\underline{Y}=\underline{y}\} + \underline{z}'\underline{z}$$

$$= E\{||\underline{X}||^2/\underline{Y}=\underline{y}\} + E\{||\underline{z}-E[\underline{X}/\underline{Y}=\underline{y}]||^2\}$$

$$- ||E\{\underline{X}/\underline{Y}=\underline{y}\}||^2$$

The only term to involve $\underline{y}$ in the previous expression is the second, and thus minimization requires that

$$\underline{z} = \hat{\underline{x}} = E\{\underline{X}/\underline{Y}=\underline{y}\}$$

The corresponding minimum of (2-19) is then

$$E\{||\underline{X}-\hat{\underline{x}}||^2/\underline{Y}=\underline{y}\} = E\{||\underline{X}||^2/\underline{Y}=\underline{y}\} - ||\hat{\underline{x}}||^2$$

The conditional mean $\hat{\underline{x}}$ will thus in general depend on the observed n-vector $\underline{y}$.

For the stochastic model described by (2-3) assume that the initial state $\underline{x}(t_0)$ is Gaussian with mean and covariance given by $E\{\underline{x}(t_0)\} = \overline{\underline{x}}_0$, $\mathrm{cov}\{\underline{x}(t_0), \underline{x}(t_0)\} = \Sigma_0$ where the noise processes $\{\underline{w}(t)\}$, $\{\underline{v}(t)\}$ are white Gaussian, with properties

$$E\{\underline{w}(t)\} = E\{\underline{v}(t)\} = \underline{0}$$

and

$$E\{\underline{w}(t)\ \underline{w}'(\tau)\} = W(t)\ \delta(t-\tau)$$

$$E\{\underline{v}(t)\ \underline{v}'(\tau)\} = V(t)\ \delta(t-\tau),\ V(t)>0$$

and such that $\underline{x}_0$, $\{\underline{w}(t)\},\{\underline{v}(t)\}$ are independent. It is then a well known result that, if $\underline{u}(t)$ is admissable, the corresponding conditional distribution of the state is Gaussian with conditional mean $\hat{\underline{x}}(t)$ and conditional covariance $\Sigma(t)$ given by [39,42] [A1]

$$\dot{\hat{\underline{x}}}(t) = A\ \hat{\underline{x}}(t) + \Sigma(t)\ C'V^{-1}(t)\ (\underline{y}(t) - C\ \hat{\underline{x}}(t))$$

$$+ B\ \underline{u}(t)\ ;\ \hat{\underline{x}}(t_0) = \overline{\underline{x}}_0 \qquad \text{(A3-3)}$$

$$\dot{\Sigma}(t) = A\ \Sigma(t) + \Sigma(t)\ A' + W(t) - \Sigma(t)\ C'V^{-1}\ C\Sigma(t)\ ;$$

$$\Sigma(t_0) = \Sigma_0 \qquad \text{(A3-4)}$$

To obtain an estimator that is relatively easy to implement, it is highly desirable that the time varying nature of the filter defined by (A3-3, A3-4) be removed. The several assumptions made in connection with the stochastic model (2-3) of section 2.1 now

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[A1] These equations are the Kalman filter equations [22, 30] modified to include the effects of the deterministic input $\underline{u}(t)$.

allow us to accomplish this. Specifying the driving and observation noises as stationary removes the time varying nature of the auto-covariance matrices W(t) and V(t), whereupon (A3-3, A3-4) become

$$\dot{\hat{\underline{x}}}(t) = A\,\hat{\underline{x}}(t) + \Sigma(t)\,C'\,V^{-1}(\underline{y}(t) - C\,\hat{\underline{x}}(t))$$

$$+ B\,\underline{u}(t)\ ;\ \underline{x}(t_o) = \overline{\underline{x}}_o \qquad \text{(A3-5)}$$

$$\dot{\Sigma}(t) = A\,\Sigma(t) + \Sigma(t)\,A' + W - \Sigma(t)\,C'\,V^{-1}\,C\,\Sigma(t)\ ;$$

$$\Sigma(t_o) = \Sigma_o \qquad \text{(A3-6)}$$

Further, since the constant system (2-3) is completely controllable and observable, for all

$$\Sigma_o \geq 0,\ [23,\ 39,\ 42] \qquad \text{A2}$$

$$\lim_{t_o \to -\infty} \Sigma(t;t_o,\Sigma_o) = \Sigma_\infty$$

where Σ_∞ is the unique solution to the algebraic Riccati equation

$$A\,\Sigma_\infty + \Sigma_\infty A' - \Sigma_\infty\,C'V^{-1}\,C\,\Sigma_\infty + W = 0;$$

$$\Sigma_\infty \geq 0 \qquad \text{(A3-7)}$$

$\hat{\underline{x}}(t)$ is thus given by the steady-state Kalman filter (modified to include the effects of the deterministic input $\underline{u}(t)$)

$$\dot{\hat{\underline{x}}}(t) = A\,\hat{\underline{x}}(t) + \Sigma_\infty\, C'\, V^{-1}\,(\underline{y}(t) - C\,\hat{\underline{x}}(t)) + B\,\underline{u}(t) \qquad \text{(A3-8)}$$

The steady-state Kalman filter (A3-8) is the best linear estimator of the state of the completely controllable and completely observable constant system (2-3) in terms of the output process $\underline{y}(\cdot)$ over the time interval $(-\infty,t)$.[A3]

[A2] To emphasize the dependence of $\Sigma(t)$, satisfying equation (2-23) on the initial time (t_0) and the initial conditions (Σ_0) let

$$\Sigma(t;\, t_0, \Sigma_0) \triangleq \Sigma(t)$$

[A3] For the case where the observed random vector $\underline{Y}$ and the random state vector $\underline{X}$ are jointly Gaussian, the conditional expectation, $E\{\underline{X}/\underline{Y}=\underline{y}\}$, is linear in $\underline{Y}$ and the unconstrained least squares estimator then coincides with the linear least squares estimator

APPENDIX FOUR

CHOOSING THE INTEGRATION ROUTINE AND THE INTEGRATION INTERVAL

Several difficulties were encountered in simulating the system described by the state-output equations

$$\dot{\underline{x}}(t) = A\ \underline{x}(t) + B\ \underline{u}(t) + \underline{w}(t)$$

$$\underline{y}(t) = x(t-\tau) + \underline{v}(t-\tau) \quad .$$

Among the more serious of these : 1. that a variable step integration routine could not be used, and 2. that the size of the fixed step integration interval was severely limited, were largely a result of the nature of the model and the simulation procedure employed by the CSMP program.

System 360/CSMP generates a random number at each iteration cycle. Quoting the CSMP manual [20], "Structure statements (these specify model dynamics and associated computations) are translated and placed into a FORTRAN subroutine called UPDATE which is executed at each iteration cycle.[A1] Then, as the integration

---

[A1] The UPDATE subroutine also contains the CALL statements for the subroutine which generates the random numbers (GAUSS(·,·,·)).

interval is reduced, the UPDATE subroutine is executed more frequently and hence, per unit of machine time, more random numbers are generated.

The noise cutoff frequency f_c is given by

$$f_c = \frac{1}{2(\Delta T)} \tag{A2-1}$$

where ΔT is the integration interval. It thus follows that whenever the integration step size changes so does the noise cutoff frequency. As a result, variable step integration routines could not be used as they would cause the noise cutoff frequency to be correlated with the shape of the solution of the simulated differential equations. For the fixed-step integration routines, the integration interval could not be made too small or the noise cutoff frequency would be so high that the regulator (with a relatively narrow bandwidth) would effectively filter it out. On the other hand, if the fixed integration interval were made too large, the resulting integration errors could give an inaccurate solution or could swamp out effects which one may have hoped to observe. Some tradeoff was thus necessary between a desirable noise cutoff frequency and allowable integration errors.

Several preliminary studies indicated that an integration step size of 0.01 second, f_c=50 Hertz, for the fourth-order Runge-Kutta integration routine was a reasonable choice.

APPENDIX FIVE

SAMPLE COMPUTER PROGRAMS USED IN THIS WORK

Some representative computer programs used in the course of this work are presented here for the reader's information. The heading of each program gives a brief description of its function.

```
      C
      C CALCULATION OF POWER SPECTRA BY
      C AVERAGING OVER 20 DATA RECORDS IN 'Z'
      C
            DIMENSION Z(20480),X(1024),Y(1024),
           1AMAGN(20,512),AMAGF(512)
            READ (8,1) U
            READ (8,1) Z
          1 FORMAT (F10.4)
            L=0
      C
      C   GENERATION OF ZERO VALUES FOR Y-ARRAY (IMAG. PART)
      C
          3 CONTINUE
            DO 2 I=1,1024
            Y(I)=0.0
          2 CONTINUE
      C
      C   GENERATION OF VALUES FOR X-ARRAY (REAL PART OF COMPLEX NO.)
      C
            N=(L*1024)+1
            M=(L+1)*1024
            J=L*1024
            DO 4 I=N,M
            K=I-J
            X(K)=Z(I)
          4 CONTINUE
      C
      C   USE FAST FOURIER TRANSFORM ON THE TIME SERIES
      C
            LOG2N=10
            IFSET=1
            CALL PS301A(LOG2N,X,Y,IFSET)
      C
      C   MAGNITUDE SQUARED OF THE FIRST HALF OF THE TRANSFORMED DATA
      C
            L=L+1
            DO 7 I=1,512
            AMAGN(L,I)=((X(I)**2)+(Y(I)**2))
          7 CONTINUE
      C
      C   TO PERFORM PREVIOUS CALCULATIONS FOR 20 DATA RECORDS
      C
            K=(20-L)
            IF (K.NE.0) GO TO 3
      C
      C   TO AVERAGE OUT VALUES OBTAINED FOR THE 20 DATA RECORDS
      C
            DO 9 K=1,512
            ASUM=0.0
            DO 10 I=1,20
            ASUM=ASUM+AMAGN(I,K)
         10 CONTINUE
            AMAGF(K)=ASUM
          9 CONTINUE
```

```
0033          DO 50 I=1,512
0034          AMAGF(I)=AMAGF(I)/20.
0035       50 CONTINUE
      C
      C    PICK OUT MAXIMUM VALUE OF THE MAGNITUDE SQUARED
      C
0036          AMAGMX=0.0
0037          DO 11 I=1,512
0038          IF (AMAGF(I).LT.AMAGMX) GO TO 12
0039          AMAGMX=AMAGF(I)
0040       12 CONTINUE
0041       11 CONTINUE
0042          WRITE (6,13) AMAGMX
0043       13 FORMAT ('-','MAXIMUM VALUE OF THE MAGNITUDE SQUARED',
             12X,E20.7)
      C
      C    TO SCALE TO MAXIMUM VALUE AND CONVERT TO DECIBELS
      C
0044          DO 23 I=1,512
0045          AMAGF(I)=AMAGF(I)/AMAGMX
0046       23 CONTINUE
0047          DO 24 I=1,512
0048          AMAGF(I)=20.*((ALOG(AMAGF(I)))/2.3026)
0049       24 CONTINUE
      C
      C    PRINT OUT SPECTRUM AND PUNCH OUT DATA CARDS
      C    FOR PLOTTING ROUTINE
      C
0050          WRITE (6,25)
0051       25 FORMAT ('-','SPECTRAL DENSITY IN DECIBELS')
0052          DO 26 I=1,512
0053          WRITE (6,27) I,AMAGF(I)
0054       27 FORMAT (1X,I3,10X,E20.7)
0055       26 CONTINUE
0056          WRITE (7,199) AMAGF
0057      199 FORMAT (8F10.4)
0058          STOP
0059          END
```

```
****CONTINUOUS SYSTEM MODELING PROGRAM****

***PROBLEM INPUT STATEMENTS***

*    SYSTEM WITH NO PLANT NOISE AND NO MEASUREMENT NOISE
*      NO KALMAN FILTER IN FEEDBACK

       LABEL STATE EQUATIONS
       LABEL MEASURED STATES
       LABEL CORRECTIVE FORCES

       INITIAL

       INCON IC1=0.0,IC2=-4.2,IC3=0.0,IC4=+2.1,IC5=0.0

       DYNAMIC

       W1=0.0
       W2=0.0
       W3=0.0
       W4=0.0
       W5=0.0
       V1=0.0
       V2=0.0
       V3=0.0
       V4=0.0
       V5=0.0

*    PLANT DYNAMICS (XDOT=AX+BU+W)

       X1=-DY1+DF1+W1
       DY1=INTGRL(IC1,X1)
       X2=DY1-DY2+W2
       DW1=INTGRL(IC2,X2)
       X3=-DY2+DF2+W3
       DY2=INTGRL(IC3,X3)
       X4=DY2-DY3+W4
       DW2=INTGRL(IC4,X4)
       X5=-DY3+DF3+W5
       DY3=INTGRL(IC5,X5)

*    OUTPUT EQUATIONS (Y=CX+V)

       Y1=DY1+V1
       Y2=DW1+V2
       Y3=DY2+V3
       Y4=DW2+V4
       Y5=DY3+V5

*    FEEDBACK FROM CONTROLLER

       DF1=-(1.263*Y1+2.494*Y2-0.819*Y3+0.668*Y4-0.444*Y5)
       DF2=-(-0.819*Y1-1.826*Y2+1.638*Y3+1.826*Y4-0.819*Y5)
       DF3=-(-0.444*Y1-0.668*Y2-0.819*Y3-2.494*Y4+1.263*Y5)

       TERMINAL

       TIMER DELT=0.010,OUTDEL=0.01,FINTIM=5.0
       PRTPLT DY1,DW1,DY2,DW2,DY3
       METHOD RKSFX
```

```
                PREPAR DY1,DW1,DY2,DW2,DY3,DF1,DF2,DF3
                END
                STOP

OUTPUT VARIABLE SEQUENCE
W1      V5      Y5      V4      Y4      V3      Y3      V2      Y2      V1
Y1      DF1     X1      DY1     W2      X2      DW1     W3      DF2     X3
DY2     W4      X4      DW2     W5      DF3     X5      DY3

 OUTPUTS     INPUTS     PARAMS   INTEGS + MEM BLKS   FORTRAN   DATA CDS
 32(500)    70(1400)    8(400)     5+   0=   5(300)   29(600)        0

           ENDJOB


18:52.22 3.603 RC=0
```

```
****CONTINUOUS SYSTEM MODELING PROGRAM****

***PROBLEM INPUT STATEMENTS***

*   SYSTEM WITH BOTH PLANT NOISE AND MEASUREMENT NOISE
*     NO KALMAN FILTER IN FEEDBACK
*        NOISE STATISTICS: W=I,V=I

      LABEL STATE EQUATIONS
      LABEL MEASURED STATES
      LABEL CORRECTIVE FORCES

      INITIAL

      INCON IC1=0.0,IC2=-4.2,IC3=0.0,IC4=+2.1,IC5=0.0

      DYNAMIC

      W1=GAUSS(1,0.00,1.)
      W2=GAUSS(3,0.00,1.)
      W3=GAUSS(5,0.00,1.)
      W4=GAUSS(7,0.00,1.)
      W5=GAUSS(9,0.00,1.)
      V1=GAUSS(11,0.00,1.)
      V2=GAUSS(13,0.00,1.)
      V3=GAUSS(15,0.00,1.)
      V4=GAUSS(17,0.00,1.)
      V5=GAUSS(19,0.00,1.)

*   PLANT DYNAMICS (XDOT=AX+BU+W)

      X1=-DY1+DF1+W1
      DY1=INTGRL(IC1,X1)
      X2=DY1-DY2+W2
      DW1=INTGRL(IC2,X2)
      X3=-DY2+DF2+W3
      DY2=INTGRL(IC3,X3)
      X4=DY2-DY3+W4
      DW2=INTGRL(IC4,X4)
      X5=-DY3+DF3+W5
      DY3=INTGRL(IC5,X5)

*   OUTPUT EQUATIONS (Y=CX+V)

      Y1=DY1+V1
      Y2=DW1+V2
      Y3=DY2+V3
      Y4=DW2+V4
      Y5=DY3+V5

*  FEEDBACK FROM CONTROLLER

      DF1=-(1.263*Y1+2.494*Y2-0.819*Y3+0.668*Y4-0.444*Y5)
      DF2=-(-0.819*Y1-1.826*Y2+1.638*Y3+1.826*Y4-0.819*Y5)
      DF3=-(-0.444*Y1-0.668*Y2-0.819*Y3-2.494*Y4+1.263*Y5)

      TERMINAL

      TIMER DELT=0.01,OUTDEL=0.01,FINTIM=5.0
      METHOD RKSFX
```

```
                  PREPAR DY1,DW1,DY2,DW2,DY3,DF1,DF2,DF3
                  END
                  STOP

OUTPUT VARIABLE SEQUENCE
W1      V5      Y5      V4      Y4      V3      Y3      V2      Y2      V1
Y1      DF1     X1      DY1     W2      X2      DW1     W3      DF2     X3
DY2     W4      X4      DW2     W5      DF3     X5      DY3

 OUTPUTS     INPUTS     PARAMS   INTEGS + MEM BLKS   FORTRAN   DATA CDS
 32(500)    70(1400)    8(400)    5+   0=   5(300)   29(600)       8

            ENDJOB


13:31.19 3.439 RC=0
```

```
                ****CONTINUOUS SYSTEM MODELING PROGRAM****

      ***PROBLEM INPUT STATEMENTS***

*       SYSTEM WITH BOTH PLANT AND MEASUREMENT NOISE
*           KALMAN FILTER IN FEEDBACK
*           NOISE STATISTICS ARE KNOWN EXACTLY
*           FILTER ASSUMES A NOISE VARIANCE OF 1.0
*           ACTUAL NOISE VARIANCE IS 1.0

        LABEL    STATE EQUATIONS
        LABEL    MEASURED STATES
        LABEL    ESTIMATED STATES
        LABEL    CORRECTIVE FORCES

        INITIAL

        INCON  IC1=0.0,IC2=-4.2,IC3=0.0,IC4=2.1,IC5=0.0
        INCON  ICE1=0.0,ICE2=-4.2,ICE3=0.0,ICE4=2.1,ICE5=0.0

        DYNAMIC

        W1=GAUSS(1,0.00,1.)
        W2=GAUSS(3,0.00,1.)
        W3=GAUSS(5,0.00,1.)
        W4=GAUSS(7,0.00,1.)
        W5=GAUSS(9,0.00,1.)
        V1=GAUSS(11,0.00,1.)
        V2=GAUSS(13,0.00,1.)
        V3=GAUSS(15,0.00,1.)
        V4=GAUSS(17,0.00,1.)
        V5=GAUSS(19,0.00,1.)

*     STATE EQUATIONS (XDOT=AX+BU+W)

        X1=-DY1+DF1+W1
        DY1=INTGRL(IC1,X1)
        X2=DY1-DY2+W2
        DW1=INTGRL(IC2,X2)
        X3=-DY2+DF2+W3
        DY2=INTGRL(IC3,X3)
        X4=DY2-DY3+W4
        DW2=INTGRL(IC4,X4)
        X5=-DY3+DF3+W5
        DY3=INTGRL(IC5,X5)

*     OUTPUT EQUATIONS (Y=CX+V)

        Y1=DY1+V1
        Y2=DW1+V2
        Y3=DY2+V3
        Y4=DW2+V4
        Y5=DY3+V5

* KALMAN FILTER
*     1.SUMBAR=SUMBAR*CTRANSPOSE*VINVERSE*Y

        SUMBR1=0.406*Y1+0.151*Y2+0.007*Y3+0.008*Y4+0.001*Y5
        SUMBR2=0.151*Y1+1.239*Y2-0.143*Y3-0.102*Y4-0.008*Y5
        SUMBR3=0.007*Y1-0.143*Y2+0.400*Y3+0.143*Y4+0.007*Y5
```

7

```
          SUMBR4=0.008*Y1-0.102*Y2+0.143*Y3+1.239*Y4-0.151*Y5
          SUMBR5=0.001*Y1-0.008*Y2+0.007*Y3-0.151*Y4+0.406*Y5

*     2.ROWDOT=(A-SUMBAR*CTRANSPOSE*VINVERSE*C)*ROW+QI

          X6=-1.406*ROW1-0.151*ROW2-0.007*ROW3-0.008*ROW4-0.001*ROW5+Q1
          ROW1=INTGRL(ICF1,X6)
          X7=0.849*ROW1-1.239*ROW2-0.857*ROW3+0.102*ROW4+0.008*ROW5+Q2
          ROW2=INTGRL(ICF2,X7)
          X8=-0.007*ROW1+0.143*ROW2-1.400*ROW3-0.143*ROW4-0.007*ROW5+Q3
          ROW3=INTGRL(ICF3,X8)
          X9=-0.008*ROW1+0.102*ROW2+0.857*ROW3-1.239*ROW4-0.849*ROW5+Q4
          ROW4=INTGRL(ICF4,X9)
          X10=-0.001*ROW1+0.008*ROW2-0.007*ROW3+0.151*ROW4-1.406*ROW5+Q5
          ROW5=INTGRL(ICF5,X10)

*     3.GI=GAMMA=OUTPUT OF FILTER

          G1=ROW1
          G2=ROW2
          G3=ROW3
          G4=ROW4
          G5=ROW5

*     4.B*U(T)=-B*LSTAR*GT          (WHERE U1=DF1,U3=DF2,U5=DF3)

          DF1=-(1.263*G1+2.494*G2-0.819*G3+0.668*G4-0.444*G5)
          DF2=-(-0.819*G1-1.826*G2+1.638*G3+1.826*G4-0.819*G5)
          DF3=-(-0.444*G1-0.668*G2-0.819*G3-2.494*G4+1.263*G5)

*     5.QI=SUMBAR+B*U(T)

          Q1=SUMBR1+DF1
          Q2=SUMBR2
          Q3=SUMBR3+DF2
          Q4=SUMBR4
          Q5=SUMBR5+DF3

          TERMINAL

          TIMER DELT=0.01,OUTDEL=0.01,FINTIM=5.0
          METHOD RKSFX
          PRTPLT   DY1,DW1,DY2,DW2,DY3
          PREPAR DY1,DW1,DY2,DW2,DY3,Y1,Y2,Y3,Y4,Y5,G1,G2,G3,G4,G5,...
          DF1,DF2,DF3
          END
          STOP

OUTPUT VARIABLE SEQUENCE
W1      G5      G4      G3      G2      G1      DF1     X1      DY1     W2
X2      DW1     W3      DF2     X3      DY2     W4      X4      DW2     W5
DF3     X5      DY3     V5      Y5      V4      Y4      V3      Y3      V2
Y2      V1      Y1      SUMBR1  Q1      X6      ROW1    SUMBR2  Q2      X7
ROW2    SUMBR3  Q3      X8      ROW3    SUMBR4  Q4      X9      ROW4    SUMBR5
Q5      X10     ROW5

 OUTPUTS    INPUTS    PARAMS  INTEGS + MEM BLKS  FORTRAN  DATA CDS
 57(500)  168(1400)  13(400)  10+  0= 10(300)    54(600)      12
```

```
                ****CONTINUOUS SYSTEM MODELING PROGRAM****

         ***PROBLEM INPUT STATEMENTS***

*    SYSTEM WITH PLANT NOISE ONLY
*      MEASUREMENT OF SYSTEM STATES EXACT; BUT WITH TIME DELAY
*      NO PREDICTOR IN FEEDBACK
*      DELAY TIME=P=0.1 SECOND

       LABEL STATE EQUATIONS
       LABEL MEASURED STATES
       LABEL CORRECTIVE FORCES

       INITIAL

       INCON IC1=0.0,IC2=-4.2,IC3=0.0,IC4=+2.1,IC5=0.0

       DYNAMIC

       W1=GAUSS(1,0.00,1.)
       W2=GAUSS(3,0.00,1.)
       W3=GAUSS(5,0.00,1.)
       W4=GAUSS(7,0.00,1.)
       W5=GAUSS(9,0.00,1.)

*    PLANT DYNAMICS (XDOT=AX+BU+W)

       X1=-DY1+DF1+W1
       DY1=INTGRL(IC1,X1)
       X2=DY1-DY2+W2
       DW1=INTGRL(IC2,X2)
       X3=-DY2+DF2+W3
       DY2=INTGRL(IC3,X3)
       X4=DY2-DY3+W4
       DW2=INTGRL(IC4,X4)
       X5=-DY3+DF3+W5
       DY3=INTGRL(IC5,X5)

*    OUTPUT EQUATIONS (Y=CX(T-P))

       Y1=DELAY(9,0.1,DY1)
       Y2=DELAY(9,0.1,DW1)
       Y3=DELAY(9,0.1,DY2)
       Y4=DELAY(9,0.1,DW2)
       Y5=DELAY(9,0.1,DY3)

*    FEEDBACK FROM CONTROLLER

       DF1=-(1.263*Y1+2.494*Y2-0.819*Y3+0.668*Y4-0.444*Y5)
       DF2=-(-0.819*Y1-1.826*Y2+1.638*Y3+1.826*Y4-0.819*Y5)
       DF3=-(-0.444*Y1-0.668*Y2-0.819*Y3-2.494*Y4+1.263*Y5)

       TERMINAL

       TIMER DELT=0.01,OUTDEL=0.01,FINTIM=5.0
       METHOD RKSFX
       PREPAR DY1,DW1,DY2,DW2,DY3,Y1,Y2,Y3,Y4,Y5,DF1,DF2,DF3
       END
       STOP
```

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```
OUTPUT VARIABLE SEQUENCE
W1      Y5      Y4      Y3      Y2      Y1      DF1     X1      DY1     W2
X2      DW1     W3      DF2     X3      DY2     W4      X4      DW2     W5
DF3     X5      DY3

 OUTPUTS    INPUTS    PARAMS  INTEGS + MEM BLKS  FORTRAN  DATA CDS
 27(500)   65(1400)   8(400)    5+   5= 10(300)   24(600)       8


          ENDJOB


12:11.06 3.425 RC=0
```

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```
FORTRAN IV G COMPILER (20.1)     MAIN               05-26-72       13:36.46      PAGE
                      C
                      C TO CALCULATE EXP(AT) WHERE
                      C 'A' IS A MATRIX AND 'T' IS A CONSTANT
                      C
 0001                      DIMENSION A(5,5),G(5,5),R(5,5),S1(5,5),S2(5,5),
                         1S3(5,5),ADD1(5,5)
                      C
                      C   READ IN VALUES FOR 'A','I'; CHECK FOR CORRECTNESS
                      C
 0002                      READ (5,105) ((A(I,J),J=1,5),I=1,5)
 0003                      READ (5,105) ((G(I,J),J=1,5),I=1,5)
 0004                  105 FORMAT (5F10.5)
 0005                      WRITE (6,100)
 0006                  100 FORMAT ('-','A(I,J) IS')
 0007                      WRITE (6,106) ((A(I,J),J=1,5),I=1,5)
 0008                      WRITE (6,101)
 0009                  101 FORMAT ('-','I(I,J) IS')
 0010                      WRITE (6,108) ((G(I,J),J=1,5),I=1,5)
 0011                  106 FORMAT (10X,5F10.3)
 0012                  108 FORMAT (10X,5F10.3)
                      C
                      C   SET ORDER OF EXPANSION; SET DELAY TIME
                      C
 0013                      N=16
 0014                      T=0.40
 0015                      DO 20 K=1,4
 0016                      T=T+0.01
                      C
 0017                      WRITE (6,300) T
 0018                  300 FORMAT (1X,'T IS',1X,F4.2)
                      C
                      C   CALCULATE SUCCESSIVE TERMS OF THE EXPANSION
                      C
 0019                      CALL SMPY(A,T,R,5,5,0)
 0020                      CALL MCPY(R,S1,5,5,0)
 0021                      CALL MCPY(S1,S2,5,5,0)
 0022                      CALL GMADD(G,S1,R,5,5)
 0023                      CALL MCPY(R,ADD1,5,5,0)
 0024                      DO 19 L=2,N
 0025                      CALL SMPY(S1,1./L,R,5,5,0)
 0026                      CALL MCPY(R,S3,5,5,0)
 0027                      CALL MPRD(S3,S2,R,5,5,0,0,5)
 0028                      CALL MCPY(R,S2,5,5,0)
                      C
 0029                      WRITE (6,200) L
 0030                  200 FORMAT ('-',15X,'TERM ',I2)
 0031                      WRITE (6,201) ((S2(I,J),J=1,5),I=1,5)
 0032                  201 FORMAT (22X,5F10.5)
                      C
                      C   ADD EACH SUCCESSIVE TERM TO PARTIAL SUM
                      C
 0033                      CALL GMADD(ADD1,S2,R,5,5)
 0034                      CALL MCPY(R,ADD1,5,5,0)
                      C
 0035                      WRITE (6,152)
```

```
0036        152 FORMAT ('-',15X,'EXP(AT) IS')
0037            WRITE (6,203) ((ADD1(I,J),J=1,5),I=1,5)
0038        203 FORMAT (22X,5F10.5)
0039         19 CONTINUE
0040         20 CONTINUE
0041            END
```

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```
                ****CONTINUOUS SYSTEM MODELING PROGRAM****

         ***PROBLEM INPUT STATEMENTS***

*    SYSTEM WITH PLANT NOISE ONLY
*      MEASUREMENT OF SYSTEM STATES EXACT; BUT WITH TIME DELAY
*      PREDICTOR ONLY IN FEEDBACK
*      DELAY TIME=P=0.1 SEC.

       LABEL STATE EQUATIONS
       LABEL MEASURED STATES
       LABEL PREDICTED STATES
       LABEL CORRECTIVE FORCES

       INITIAL

       INCON IC1=0.0,IC2=-4.2,IC3=0.0,IC4=2.1,IC5=0.0
       INCON ICD1=0.0,ICD2=-4.2,ICD3=0.0,ICD4=2.1,ICD5=0.0
       PARAMETER EX11=0.9049,EX12=0.0,EX13=0.0,EX14=0.0,EX15=0.0,...
       EX21=0.0952,EX22=1.0,EX23=-0.0952,EX24=0.0,EX25=0.0,...
       EX31=0.0,EX32=0.0,EX33=0.9048 ,EX34=0.0,EX35=0.0,...
       EX41=0.0,EX42=0.0,EX43=0.0952,EX44=1.0,EX45=-0.0952,...
       EX51=0.0,EX52=0.0,EX53=0.0,EX54=0.0,EX55=0.9048

       DYNAMIC

       W1=GAUSS(1,0.00,1.)
       W2=GAUSS(3,0.00,1.)
       W3=GAUSS(5,0.00,1.)
       W4=GAUSS(7,0.00,1.)
       W5=GAUSS(9,0.00,1.)

*    STATE EQUATIONS (XDOT=AX+BU+W)

       X1=-DY1+DF1+W1
       DY1=INTGRL(IC1,X1)
       X2=DY1-DY2+W2
       DW1=INTGRL(IC2,X2)
       X3=-DY2+DF2+W3
       DY2=INTGRL(IC3,X3)
       X4=DY2-DY3+W4
       DW2=INTGRL(IC4,X4)
       X5=-DY3+DF3+W5
       DY3=INTGRL(IC5,X5)

*    OUTPUT EQUATIONS (Y=CX(T-P))

       Y1=DELAY(9,0.1,DY1)
       Y2=DELAY(9,0.1,DW1)
       Y3=DELAY(9,0.1,DY2)
       Y4=DELAY(9,0.1,DW2)
       Y5=DELAY(9,0.1,DY3)

* LEAST MEAN-SQUARED PREDICTOR
*    1.DETERMINISTIC PART OF PLANT MODEL

       X11=-Z1+DF1
       Z1=INTGRL(ICD1,X11)
       X12=Z1-Z3
       Z2=INTGRL(ICD2,X12)
```

```
      X13=-Z3+DF2
      Z3=INTGRL(ICD3,X13)
      X14=Z3-Z5
      Z4=INTGRL(ICD4,X14)
      X15=-Z5+DF3
      Z5=INTGRL(ICD5,X15)

*   2.DELAY OF DETERMINISTIC MODEL STATES

      YU1=DELAY(9,0.1,Z1)
      YU2=DELAY(9,0.1,Z2)
      YU3=DELAY(9,0.1,Z3)
      YU4=DELAY(9,0.1,Z4)
      YU5=DELAY(9,0.1,Z5)

*   3.R(T)=Y(T)-YU(T)

      SI1=Y1-YU1
      SI2=Y2-YU2
      SI3=Y3-YU3
      SI4=Y4-YU4
      SI5=Y5-YU5

*   4.EXR=EXP(AP)*R(T)

      EXR1=EX11*SI1+EX12*SI2+EX13*SI3+EX14*SI4+EX15*SI5
      EXR2=EX21*SI1+EX22*SI2+EX23*SI3+EX24*SI4+EX25*SI5
      EXR3=EX31*SI1+EX32*SI2+EX33*SI3+EX34*SI4+EX35*SI5
      EXR4=EX41*SI1+EX42*SI2+EX43*SI3+EX44*SI4+EX45*SI5
      EXR5=EX51*SI1+EX52*SI2+EX53*SI3+EX54*SI4+EX55*SI5

*   5.XHAT(T)=EXR+Z

      D1=EXR1+Z1
      D2=EXR2+Z2
      D3=EXR3+Z3
      D4=EXR4+Z4
      D5=EXR5+Z5

*   USTAR(T)=-LSTAR*XHAT(T)

      DF1=-(1.263*D1+2.494*D2-0.819*D3+0.668*D4-0.444*D5)
      DF2=-(-0.819*D1-1.826*D2+1.638*D3+1.826*D4-0.819*D5)
      DF3=-(-0.444*D1-0.668*D2-0.819*D3-2.494*D4+1.263*D5)

      TERMINAL

      TIMER DELT=0.010,OUTDEL=0.01,FINTIM=5.0
      METHOD RKSFX
      PREPAR DY1,DW1,DY2,DW2,DY3,Y1,Y2,Y3,Y4,Y5,D1,D2,D3,D4,D5,...
      DF1,DF2,DF3
      END
      STOP
```

```
OUTPUT VARIABLE SEQUENCE
W1     SI5    SI4    SI3    SI2    SI1    EXR5   D5     EXR4   D4
EXR3   D3     EXR2   D2     YU5    Y5     YU4    Y4     YU3    Y3
YU2    Y2     YU1    Y1     EXR1   D1     DF1    X1     DY1    W2
X2     DW1    W3     DF2    X3     DY2    W4     X4     DW2    W5
DF3    X5     DY3    X11    Z1     X12    Z2     X13    Z3     X14
```

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```
Z4      X15     Z5

 OUTPUTS     INPUTS    PARAMS  INTEGS + MEM BLKS  FORTRAN  DATA CDS
 57(500)  160(1400)  38(400)  10+ 10= 20(300)   54(600)      16

          ENDJOB

10:08.16 6.33 RC=0
```

```
         ****CONTINUOUS SYSTEM MODELING PROGRAM****

      ***PROBLEM INPUT STATEMENTS***

*   SYSTEM WITH PLANT NOISE,MEASUREMENT NOISE,AND FEEDBACK TIME DELAY
*      POSITION DEVIATIONS ALONE INCUR A COST
*      TIME DELAY=0.3 SEC.; NOISE VARIANCE=1.0

       LABEL    STATE EQUATIONS
       LABEL    MEASURED STATES
       LABEL    ESTIMATED STATES
       LABEL    PREDICTED STATES
       LABEL    CORRECTIVE FORCES

       INITIAL

       INCON IC1=0.0,IC2=0.0,IC3=0.0,IC4=0.0,IC5=0.
       INCON ICE1=0.0,ICE2=0.0,ICE3=0.0,ICE4=0.0,ICE5=0.0
       INCON ICD1=0.0,ICD2=0.0,ICD3=0.0,ICD4=0.0,ICD5=0.0
       PARAMETER EX11=0.7408,EX12=0.0,EX13=0.0,EX14=0.0,EX15=0.0,...
       EX21=0.2592,EX22=1.0000,EX23=-0.2592,EX24=0.0,EX25=0.0,...
       EX31=0.0,EX32=0.0,EX33=0.7408,EX34=0.0,EX35=0.0,...
       EX41=0.0,EX42=0.0,EX43=0.2592,EX44=1.0000,EX45=-0.2592,...
       EX51=0.0,EX52=0.0,EX53=0.0,EX54=0.0,EX55=0.7408

       DYNAMIC

       W1=GAUSS(1,0.0,1.0)
       W2=GAUSS(3,0.0,1.0)
       W3=GAUSS(5,0.0,1.0)
       W4=GAUSS(7,0.0,1.0)
       W5=GAUSS(9,0.0,1.0)
       V1=GAUSS(11,0.0,1.0)
       V2=GAUSS(13,0.0,1.0)
       V3=GAUSS(15,0.0,1.0)
       V4=GAUSS(17,0.0,1.0)
       V5=GAUSS(19,0.0,1.0)

*   STATE EQUATIONS (XDOT=AX+BU+W)

       X1=-DY1+DF1+W1
       DY1=INTGRL(IC1,X1)
       X2=DY1-DY2+W2
       DW1=INTGRL(IC2,X2)
       X3=-DY2+DF2+W3
       DY2=INTGRL(IC3,X3)
       X4=DY2-DY3+W4
       DW2=INTGRL(IC4,X4)
       X5=-DY3+DF3+W5
       DY3=INTGRL(IC5,X5)

*   CORRUPTION OF PLANT STATES BY NOISE

       YZ1=DY1+V1
       YZ2=DW1+V2
       YZ3=DY2+V3
       YZ4=DW2+V4
       YZ5=DY3+V5

*   DELAY OF CORRUPTED PLANT STATES
```

```
      Y1=DELAY(29,0.3,YZ1)
      Y2=DELAY(29,0.3,YZ2)
      Y3=DELAY(29,0.3,YZ3)
      Y4=DELAY(29,0.3,YZ4)
      Y5=DELAY(29,0.3,YZ5)

* KALMAN FILTER
*    1.SUMBAR=SUMBAR*CTRANSPOSE*VINVERSE*Y

      SUMBR1=0.406*Y1+0.151*Y2+0.007*Y3+0.008*Y4+0.001*Y5
      SUMBR2=0.151*Y1+1.239*Y2-0.143*Y3-0.102*Y4-0.008*Y5
      SUMBR3=0.007*Y1-0.143*Y2+0.400*Y3+0.143*Y4+0.007*Y5
      SUMBR4=0.008*Y1-0.102*Y2+0.143*Y3+1.239*Y4-0.151*Y5
      SUMBR5=0.001*Y1-0.008*Y2+0.007*Y3-0.151*Y4+0.406*Y5

*    2.B*U(T-P)=-B*LSTAR*EXP(-SP)

      DF1D=DELAY(29,0.3,DF1)
      DF2D=DELAY(29,0.3,DF2)
      DF3D=DELAY(29,0.3,DF3)

*    3.QI=SUMBAR+B*U(T-P)

      Q1=SUMBR1+DF1D
      Q2=SUMBR2
      Q3=SUMBR3+DF2D
      Q4=SUMBR4
      Q5=SUMBR5+DF3D

*    4.ROWDOT=(A-SUMBAR*CTRANSPOSE*VINVERSE*C)*ROW+QI

      X6=-1.406*ROW1-0.151*ROW2-0.007*ROW3-0.008*ROW4-0.001*ROW5+Q1
      ROW1=INTGRL(ICE1,X6)
      X7=0.849*ROW1-1.239*ROW2-0.857*ROW3+0.102*ROW4+0.008*ROW5+Q2
      ROW2=INTGRL(ICE2,X7)
      X8=-0.007*ROW1+0.143*ROW2-1.400*ROW3-0.143*ROW4-0.007*ROW5+Q3
      ROW3=INTGRL(ICE3,X8)
      X9=-0.008*ROW1+0.102*ROW2+0.857*ROW3-1.239*ROW4-0.849*ROW5+Q4
      ROW4=INTGRL(ICE4,X9)
      X10=-0.001*ROW1+0.008*ROW2-0.007*ROW3+0.151*ROW4-1.406*ROW5+Q5
      ROW5=INTGRL(ICE5,X10)

*    5.GI=GAMMA=OUTPUT OF FILTER

      G1=ROW1
      G2=ROW2
      G3=ROW3
      G4=ROW4
      G5=ROW5

* LEAST MEAN-SQUARED PREDICTOR
*    1.DETERMINISTIC PART OF PLANT MODEL

      X11=-Z1+DF1
      Z1=INTGRL(ICD1,X11)
      X12=Z1-Z3
      Z2=INTGRL(ICD2,X12)
      X13=-Z3+DF2
      Z3=INTGRL(ICD3,X13)
      X14=Z3-Z5
```

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```
          Z4=INTGRL(ICD4,X14)
          X15=-Z5+DF3
          Z5=INTGRL(ICD5,X15)

*    2.DELAY OF DETERMINISTIC MODEL STATES

          YU1=DELAY(29,0.30,Z1)
          YU2=DELAY(29,0.30,Z2)
          YU3=DELAY(29,0.30,Z3)
          YU4=DELAY(29,0.30,Z4)
          YU5=DELAY(29,0.30,Z5)

*    3.R(T)=GI(T)-YUI(T)

          SI1=G1-YU1
          SI2=G2-YU2
          SI3=G3-YU3
          SI4=G4-YU4
          SI5=G5-YU5

*    4.EXR=EXP(AP)*R(T)

          EXR1=EX11*SI1+EX12*SI2+EX13*SI3+EX14*SI4+EX15*SI5
          EXR2=EX21*SI1+EX22*SI2+EX23*SI3+EX24*SI4+EX25*SI5
          EXR3=EX31*SI1+EX32*SI2+EX33*SI3+EX34*SI4+EX35*SI5
          EXR4=EX41*SI1+EX42*SI2+EX43*SI3+EX44*SI4+EX45*SI5
          EXR5=EX51*SI1+EX52*SI2+EX53*SI3+EX54*SI4+EX55*SI5

*    5.XHAT(T)=EXR+Z

          D1=EXR1+Z1
          D2=EXR2+Z2
          D3=EXR3+Z3
          D4=EXR4+Z4
          D5=EXR5+Z5

*    USTAR(T)=-LSTAR*XHAT(T)

          DF1=-(1.263*D1+2.494*D2-0.819*D3+0.668*D4-0.444*D5)
          DF2=-(-0.819*D1-1.826*D2+1.638*D3+1.826*D4-0.819*D5)
          DF3=-(-0.444*D1-0.668*D2-0.819*D3-2.494*D4+1.263*D5)

          TERMINAL

          TIMER DELT=0.010,OUTDEL=0.01,FINTIM=5.0
          METHOD RKSFX
          PREPAR DY1,DW1,DY2,DW2,DY3,Y1,Y2,Y3,Y4,Y5,G1,G2,G3,G4,G5,...
          D1,D2,D3,D4,D5,DF1,DF2,DF3
          END
          STOP

OUTPUT VARIABLE SEQUENCE
W1      G5      SI5     G4      SI4     G3      SI3     G2      SI2     G1
SI1     EXR5    D5      EXR4    D4      EXR3    D3      EXR2    D2      YU5
YU4     YU3     YU2     YU1     EXR1    D1      DF1     X1      DY1     W2
X2      DW1     W3      DF2     X3      DY2     W4      X4      DW2     Y5
DF3     X5      DY3     DF1D    V5      YZ5     Y5      V4      YZ4     Y4
V3      YZ3     Y3      V2      YZ2     Y2      V1      YZ1     Y1      SUMBR1
Q1      X6      ROW1    SUMBR2  Q2      X7      ROW2    DF2D    SUMBR3  Q3
X8      ROW3    SUMBR4  Q4      X9      ROW4    DF3D    SUMBR5  Q5      X10
```

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```
ROWS    X11    Z1     X12    Z2     X13    Z3     X14    Z4     X15
Z5

 OUTPUTS    INPUTS    PARAMS  INTEGS + MEM BLKS  FORTRAN  DATA CDS
 95(500)  251(1400)  43(400)  15+ 13= 28(300)   92(600)     18

         ENDJOB

10:50.35 0.431 RC=0
```

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